Exercise 1. ***Relaxation time approximation***

In this exercise we will show that the so-called single-relaxation-time approximation,

$$\left(\frac{\partial f(\boldsymbol{k})}{\partial t}\right)_{\text{coll}} = -\int \frac{\mathrm{d}^d k'}{(2\pi)^d} W(\boldsymbol{k},\boldsymbol{k}')[f(\boldsymbol{k}) - f(\boldsymbol{k}')] \quad \longrightarrow \quad -\frac{f(\boldsymbol{k}) - f_0(\boldsymbol{k})}{\tau}, \tag{1}$$

is a true solution to the Boltzmann equation under certain conditions.

We consider a spatially homogeneous two-dimensional metal with an isotropic Fermi surface ($\varepsilon_{\boldsymbol{k}} = \hbar^2 k^2/2m$) at zero temperature. The impurity scattering responsible for a finite resistivity is described by a delta potential in real space,

$$V_{\text{imp}}(\boldsymbol{r}) = V_0\,\delta(\boldsymbol{r}). \tag{2}$$

The system is subject to a homogeneous and time-independent electric field along the x-axis.

a) Show that the transition rates $W(\boldsymbol{k},\boldsymbol{k}')$ for the impurity potential (2) are constant and non-zero only for scattering events conserving the energy of the incoming state.

b) Write down the static Boltzmann transport equation for this setup in the form

$$\text{“drift-term”} = \text{“collision-integral”} \tag{3}$$

and take advantage of the zero-temperature limit and the symmetries of the system to eliminate all but angular variables.

c) In a case with only angular dependence, it turns out to be useful to expand the drift term $\boldsymbol{\nabla}_{\boldsymbol{k}} f \cdot (e\boldsymbol{E})$ and $\delta f = f - f_0$ in Fourier modes

$$\delta f = \sum_l f_l\, e^{il\varphi}, \qquad \boldsymbol{\nabla}_{\boldsymbol{k}} f \cdot (e\boldsymbol{E}) = \sum_l d_l\, e^{il\varphi}. \tag{4}$$

Rewrite the Boltzmann equation as a set of algebraic equations for the coefficients in the expansion (4)

$$d_m = \sum_n L_{m,n} f_n. \tag{5}$$

d) What are the eigenvalues of the so-called collision operator $L_{m,n}$ and what is their meaning? How can one interpret vanishing eigenvalues?

e) Find a solution to equation (5) and compare δf to the single-relaxation-time approximation, equation (1).

Exercise 2. ***Residual Resistivity of Copper***

Recapitulate Section 6.3.1 and try to find an explanation for the data in Table 2. What is the major reason for the increase of the resisitivity?

Impurity	Resistivity (per 1% of impurity atoms) $\rho/(10^{-8}\Omega\text{m})$
Be	0.64
Mg	0.6
B	1.4
Al	1.2
In	1.2
Si	3.2
Ge	3.7
Sn	2.8
As	6.5
Sb	5.4

Table 1: Residual resistivity of Cu for different impurities (From Landolt-Börnstein Tables, Vol 15, Springer, 1982)

Hint: In analogy to doped semiconductors, assume that an impurity atom will adjust itself corresponding to its neighbourhood. This means that it rejects all electrons from its outer shells in order to match the occupancy of its surrounding atoms. Hence the impurity is left with an effective nuclear charge Z. The explanation of the above table lies in this fact.