Sheet 6

Due date: 11 April 2014

Exercise 1 [*Energy momentum tensor*]: The energy momentum tensor of the electromagnetic field is defined by

$$T^{\mu\nu} = \frac{1}{4\pi k}\left[F^{\mu}{}_{\sigma}\,F^{\sigma\nu} - \frac{1}{4}F_{\rho\sigma}F^{\sigma\rho}g^{\mu\nu}\right] .$$

(i) Write out the energy-momentum tensor in terms of the electromagnetic fields.

(ii) Show that the conservation law reads

$$\frac{\partial}{\partial x^{\nu}}T^{\mu\nu} = -f^{\mu} ,$$

where f^{μ} is the force density

$$f^{\mu} = \left(\frac{1}{c}\mathbf{j}\cdot\mathbf{E},\ \rho\,\mathbf{E} + \frac{1}{c}\mathbf{j}\wedge\mathbf{B}\right) .$$

(iii) By integrating this equation over a finite volume V and using the divergence theorem, show that

$$\frac{d}{dt}\int_V d^3\mathbf{x}\,\frac{1}{c}T^{\mu 0} = -\int_{\partial V}\sum_{k=1}^{3}T^{\mu k}dS_k - \int_V d^3\mathbf{x} f^{\mu} .$$

Interpret the different terms as energy and momentum density, and energy and momentum current, respectively.

Exercise 2 [*Invariants*]:

(i) Show that the quantities

$$I_1 = F_{\mu\nu}F^{\mu\nu} \qquad \text{and} \qquad I_2 = \epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$$

are Lorentz invariant.

(ii) Express I_1 and I_2 in terms of the electromagnetic fields.

(iii) Suppose that in a particular inertial system $\mathbf{E}$ and $\mathbf{B}$ are orthogonal to one another. Show that this is then true in any inertial system.

(iv) In an inertial system we have $\mathbf{E} = 0$ and $\mathbf{B} \neq 0$. Is there then an intertial system in which $\mathbf{B} = 0$?