Sheet 7

Due date: 18 April 2014

Exercise 1 [*Hamilton formalism for electrodynamics*]: The Lagrange function describing a massive charged particle in an external electromagnetic field is

$$L(\mathbf{x}, \mathbf{v}, t) = -mc^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}} - e\Big(\Phi - \frac{\mathbf{v}}{c}\cdot\mathbf{A}\Big)\ .$$

(i) Define the conjugate momentum, and find the corresponding Hamilton function.

(ii) Determine the Hamiltonian equations, and show that they are equivalent to the relativistic equations of motion

$$\frac{d}{dt}\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} = e\Big(\mathbf{E} + \frac{1}{c}(\mathbf{v}\wedge\mathbf{B})\Big)\ .$$

Exercise 2 [*Poisson brackets*]: On the space of functions defined on phase space, the Poisson bracket is defined as

$$\{F, G\} = \sum_{\alpha=1}^{f}\left(\frac{\partial F}{\partial q^\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial q^\alpha}\right)\ .$$

(i) Show that the Poisson bracket satisfies the Jacobi identities

$$\{F_1, \{F_2, F_3\}\} + \{F_2, \{F_3, F_1\}\} + \{F_3, \{F_1, F_1\}\} = 0\,.$$

(ii) Assume that the Hamiltonian of a system is not explicitly time-dependent. Then it follows from Hamilton's equations of motion that

$$\frac{d}{dt}F(q(t), p(t)) = \{H, F\}\,.$$

Show that if F and G are conserved quantities then so is the Poisson bracket $\{F, G\}$.

(iii) Let $\mathbf{L}$ be the angular momentum $\mathbf{L} = \mathbf{x}\wedge\mathbf{p}$ and $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ the three unit vectors of $\mathbb{R}^3$. From the fact that $\mathbf{L}\cdot\mathbf{e}_1$ and $\mathbf{L}\cdot\mathbf{e}_2$ are conserved deduce that $\mathbf{L}\cdot\mathbf{e}_3$ is a conserved quantity.