Sheet 8

Due date: 9 May 2014

Exercise 1 [*Probability current*]: Show that the wave-function

$$\Psi_k(x,t) = \left(Ae^{\frac{ikx}{\hbar}} + Be^{-\frac{ikx}{\hbar}}\right) e^{-\frac{iE_k}{\hbar}t} ,$$

where A, B and k are constants and $E_k = \frac{k^2}{2m}$, is a solution of the time-dependent Schrödinger equation with $V(x) = 0$. In addition, compute the probability current corresponding to $\Psi_k(x,t)$ and show that it equals

$$j(x,t) = \left(|A|^2 - |B|^2\right) \frac{k}{m} .$$

Exercise 2 [*Galilei invariance of the Schrödinger equation*]: Assuming that $\Psi(x,t)$ is a solution of the time-dependent Schrödinger equation for a free particle, i. e.

$$i\hbar\partial_t\Psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2\Psi(x,t) ,$$

show that

$$\Psi_u(x,t) = \Psi(x-ut,t)\exp\left[\frac{im}{\hbar}ux - \frac{im}{2\hbar}u^2 t\right]$$

is also a solution of the time-dependent Schrödinger equation. Here u is an arbitrary constant. Interpret your finding in terms of the Galilei symmetry of the problem.

Exercise 3 [*Particle in an expanding box*]: Consider a particle moving in the box $[0,a]$, and assume that it is, for $t < t_0$, in the lowest (stationary) energy eigenstate

$$\psi_0(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) .$$

At $t = t_0$ the box is enlarged instantaneously to the interval $[0,2a]$. Compute the wave function of the particle for $t > t_0$, and show that it is in a superposition of eigenstates with energy

$$E_n = \frac{n^2\pi^2\hbar^2}{8ma^2},\ n = 2 \quad \text{or} \quad n = 1,3,5\dots$$

Finally, deduce that the probability that the particle possesses the same energy as before the enlargement of the box is 1/2.