Exercise Sheet VI

Hand in by 19.11.2008

Problem 1 [*Reparametrisations generated by Virasoro operators.*]:

(a) Consider the string at $\tau = 0$. Which of the combinations

$$L_m^{\perp} - L_{-m}^{\perp} \qquad \text{and} \qquad i(L_m^{\perp} + L_{-m}^{\perp})$$

reparametrises the σ coordinate of the string while keeping $\tau = 0$? When $\tau = 0$ is preserved, the world-sheet reparametrisation is actually a *string* reparametrisation. Show that the generators of these reparametrisations form a subalgebra of the Virasoro algebra.

(b) Describe the general *world-sheet* reparametrisation that leaves the midpoint $\sigma = \pi/2$ of the $\tau = 0$ open string fixed. Express this reparametrisation using an infinite set of constrained parameters.

Problem 2 [*Reparametrisations and constraints.*]:

(a) Verify that the reparametrisation parameters in

$$\xi_m^{\tau}(\tau, \sigma) = -ie^{im\tau} \cos m\sigma,$$
$$\xi_m^{\sigma}(\tau, \sigma) = e^{im\tau} \sin m\sigma.$$

satisfy the relations (omitting the subscript m for convenience)

$$\dot{\xi}^{\tau} = \xi^{\sigma\prime}, \qquad \dot{\xi}^{\sigma} = \xi^{\tau\prime}.$$

(b) Think of the reparametrisations

$$\tau \mapsto \tau + \epsilon \xi_m^{\tau}(\tau, \sigma),$$
$$\sigma \mapsto \sigma + \epsilon \xi_m^{\sigma}(\tau, \sigma),$$

generated by the Virasoro operators as a change of coordinates

$$\tau' = \tau + \epsilon \xi^{\tau}(\tau, \sigma), \qquad \sigma' = \sigma + \xi^{\sigma}(\tau, \sigma).$$

Note that for infinitesimal ϵ the above equations also imply that

$$\tau = \tau' - \epsilon \xi^{\tau}(\tau', \sigma'), \qquad \sigma = \sigma' - \epsilon \xi^{\sigma}(\tau', \sigma').$$

Show that the classical constraints

$$\partial_{\tau} X \cdot \partial_{\sigma} X = 0, \qquad (\partial_{\tau} X)^2 + (\partial_{\sigma} X)^2 = 0,$$

assumed to hold in (τ, σ) coordinates, also hold in (τ', σ') coordinates (to order ϵ).

Problem 3 [*Unoriented open strings.*]: The open string $X^\mu(\tau,\sigma)$, with $\sigma \in [0,\pi]$ and fixed τ, is a parametrised curve in spacetime. The orientation of a string is the direction of increasing σ on this curve.

(a) Consider now the open string $X^\mu(\tau, \pi-\sigma)$ at the same time τ. How is this second string related to the first string above? How are their endpoints and orientations related? Make a rough sketch showing the original string as a continuous curve in spacetime, and the second string as a dashed curve in spacetime.

Assume there is an orientation reversing *twist* operator Ω such that

$$\Omega X^I(\tau,\sigma)\Omega^{-1} = X^I(\tau,\pi-\sigma). \tag{1}$$

Moreover, assume that

$$\Omega x_0^- \Omega^{-1} = x_0^-, \qquad\qquad \Omega p^+ \Omega^{-1} = p^+.$$

(b) Use the open string oscillator expansion

$$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I \cos n\sigma e^{-in\tau}$$

to calculate

$$\Omega x_0^I\Omega^{-1}, \quad \Omega\alpha_0^I\Omega^{-1}, \quad \text{and} \quad \Omega\alpha_n^I\Omega^{-1}\ (n\neq 0).$$

(c) Show that $\Omega X^-(\tau,\sigma)\Omega^{-1} = X^-(\tau,\pi-\sigma)$. Since $\Omega X^+(\tau,\sigma)\Omega^{-1} = X^+(\tau,\pi-\sigma)$, equation (1) actually holds for all string coordinates. We say that orientation reversal is a symmetry of open string theory because it leaves the open string Hamiltonian H invariant: $\Omega H\Omega^{-1} = H$. Explain why this is true.

(d) Assume that the ground states are twist invariant:

$$\Omega|p^+,\vec{p}_T\rangle = \Omega^{-1}|p^+,\vec{p}_T\rangle = |p^+,\vec{p}_T\rangle.$$

List the open sting states for $N^\perp \leq 3$, and give their twist eigenvalues. Prove that, in general,

$$\Omega = (-1)^{N^\perp}.$$

(e) A state is said to be *unoriented* if it is invariant under twist. If you are commissioned to build a theory of unoriented open strings, which of the states in part (d) would you have to discard? In general, which levels of the original string state space must be discarded?

Problem 4 [*State counting.*]: The Fock space of the open string $\mathcal{H}$ is generated from the ground state $|p^+,\vec{p}_T\rangle$ with

$$\alpha_n^I|p^+,\vec{p}_T\rangle = 0 \qquad n > 0$$

by the action of the creation operators α^I_{-n} with $n > 0$. Show that

$$\mathrm{Tr}_{\mathcal{H}}\left(q^{\alpha' M^2}\right) = \frac{1}{\eta^{24}(q)} \ ,$$

where $\eta(q)$ is the Dedekind eta function

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \ ,$$

and M^2 is the mass-squared operator

$$M^2 = \frac{1}{\alpha'} \left(-1 + \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I \right) .$$

Problem 5 [*Lorentz covariance.*]: The M^{-I} quantum Lorentz generator is given by

$$\begin{aligned} M^{-I} =& x_0^- p^I - \frac{1}{4\alpha' p^+} \left(x_0^I \left(L_0^\perp + a \right) + \left(L_0^\perp + a \right) x_0^I \right) \\ & - \frac{i}{\sqrt{2\alpha'} p^+} \sum_{n=1}^{\infty} \frac{1}{n} \left(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp \right) . \end{aligned}$$

Calculate the expectation value of the commutator

$$\langle p^+, \vec{0} | \alpha_m^I [M^{-I}, M^{-J}] \alpha_{-m}^J | p^+, \vec{0} \rangle$$

and show that it equals

$$-\frac{m^2}{\alpha' (p^+)^2} \left[m \left(1 - \frac{D-2}{24} \right) + \frac{1}{m} \left(\frac{D-2}{24} + a \right) \right] .$$