Exercise Sheet VII

Hand in by 26.11.2008

Problem 1 [*Commutation relations for oscillators.*]: The set of functions $e^{in\sigma}$, with $n \in \mathbb{Z}$, is complete on the interval $\sigma \in [0, 2\pi]$ i.e.

$$\delta(\sigma - \sigma') = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in(\sigma - \sigma')}. \tag{1}$$

(a) Compute explicitly the commutator $[X^I(\tau, \sigma), \mathcal{P}^{\tau J}(\tau, \sigma')]$ using the mode expansions of X and $\mathcal{P}$ and the commutation relations

$$[\alpha^I_m, \alpha^J_n] = m\delta_{m+n,0}\eta^{IJ},$$
$$[\bar{\alpha}^I_m, \bar{\alpha}^J_n] = m\delta_{m+n,0}\eta^{IJ},$$
$$[\alpha^I_m, \bar{\alpha}^J_n] = 0.$$

Use equation (1) to confirm that the expected answer

$$[X^I(\tau, \sigma), \mathcal{P}^{\tau J}(\tau, \sigma')] = i\delta(\sigma - \sigma')\eta^{IJ}$$

emerges.

(b) Prove the zero mode commutations relations

$$[x^I_0, \alpha^J_0] = [x^I_0, \bar{\alpha}^J_0] = i\sqrt{\frac{\alpha'}{2}}\eta^{IJ},$$

starting with a derivation of

$$[x^I_0 + \sqrt{2\alpha'}\alpha^I_0, \dot{X}^J(\tau, \sigma')] = i\alpha'\eta^{IJ}.$$

Problem 2 [*Action of $L^\perp_0 - \bar{L}^\perp_0$.*]:

(a) Prove that equation

$$e^{-iP\sigma_0} X^I(\tau, \sigma) e^{iP\sigma_0} = X^I(\tau, \sigma + \sigma_0),$$

holds for finite σ_0. You may find it useful to define $f(\sigma_0) = e^{-iP\sigma_0} X^I(\tau, \sigma) e^{iP\sigma_0}$ and to calculate multiple derivatives of f, evaluated at $\sigma_0 = 0$.

(b) Explain why

$$e^{-iP\sigma_0}(\dot{X}^I \pm X^{I\prime})(\tau, \sigma) e^{iP\sigma_o} = (\dot{X}^I \pm X^{I\prime})(\tau, \sigma + \sigma_0). \tag{2}$$

(c) Use equation (2) to calculate $e^{-iP\sigma_0}\alpha^I_n e^{iP\sigma_0}$ and $e^{-iP\sigma_0}\bar{\alpha}^I_n e^{iP\sigma_0}$. In doing so, you are finding the action of a σ translation on the oscillators.

(d) Consider the state

$$|U\rangle = \alpha_{-m}^{I}\bar{\alpha}_{-n}^{J}|p^{+},\vec{p}_{T}\rangle, \quad m,n>0.$$

Use the the results of (c) to calculate $e^{-iP\sigma_0}|U\rangle$. What is the condition that makes the state $|U\rangle$ invariant under σ translations?

Problem 3 [*Unoriented closed strings.*]: This is the closed string version of problem 3 exercise VI. The closed string $X^{\mu}(\tau,\sigma)$ with $\sigma\in[0,2\pi]$ and fixed τ is a parametrised closed curve in spacetime. The orientation of a string is the direction of increasing σ on this curve.

(a) Consider now the closed string $X^{\mu}(\tau,2\pi-\sigma)$ with the same τ as above. How is this second string related to the first above? How are their orientations related? Make a rough sketch, showing the original strings as a continuous line and the second string as a dashed line.

Assume there is an orientation reversing *twist* operator Ω such that

$$\Omega X^{I}(\tau,\sigma)\Omega^{-1} = X^{I}(\tau,2\pi-\sigma). \tag{3}$$

Moreover, assume that

$$\Omega x_0^{-}\Omega^{-1} = x_0^{-}, \quad \Omega p^{+}\Omega^{-1} = p^{+}.$$

(b) Use the closed string oscillator expansion

$$X^{\mu}(\tau,\sigma) = x_0^{\mu} + \sqrt{2\alpha'}\alpha_0^{\mu}\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\alpha_n^{\mu}e^{in\sigma} + \bar{\alpha}_n^{\mu}e^{-in\sigma})$$

to calculate

$$\Omega x_0^{I}\Omega^{-1},\ \Omega\alpha_0^{I}\Omega^{-1},\ \Omega\alpha_n^{I}\Omega^{-1} \text{ and } \Omega\bar{\alpha}_n^{I}\Omega^{-1}.$$

(c) Show that $\Omega X^{-}(\tau,\sigma)\Omega^{-1} = X^{-}(\tau,2\pi-\sigma)$. Since $\Omega X^{+}(\tau,\sigma)\Omega^{-1} = X^{+}(\tau,2\pi-\sigma)$, equation (3) actually holds for all string coordinates. We say that oriented reversal is a symmetry of closed string theory because it leaves the closed string Hamiltonian H invariant: $\Omega H\Omega^{-1} = H$. Explain why this is true.

(d) Assume that the ground states are twist invariant. List the closed string states for $N^{\perp}\leq 2$, and give their twist eigenvalues. If you were commissioned to build a theory of *unoriented* closed strings, which of the states would you have to discard? What are the massless fields of unoriented closed string theory?