General relativity, exercise sheet 1.

HS 08 *Due:* Fri, October 3, 2008

1. Stereographic projection

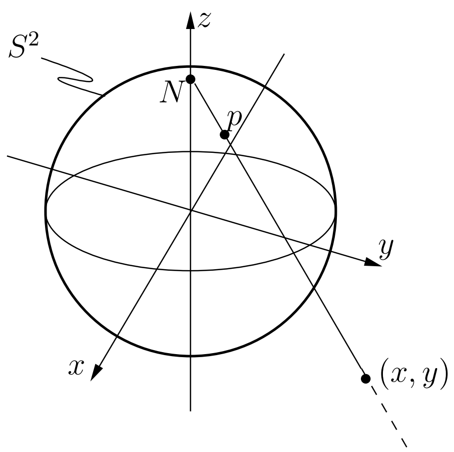
Consider the sphere

$$S^2 = \{p = (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

and its covering $S^2 = U_+ \cup U_-$ by the two open sets

$$U_\pm = S^2 \setminus \{(0, 0, \pm 1)\}$$

obtained by removing the north, resp. south pole from S^2. The stereographic projection shown in the figure provides a chart for U_+ with coordinate patch $\mathbb{R}^2 \ni (x, y)$; similarly there is one for U_- with patch $\mathbb{R}^2 \ni (\bar{x}, \bar{y})$. On which subset of $\mathbb{R}^2$ is the transition function $(x, y) \mapsto (\bar{x}, \bar{y})$ defined? Compute the function.



2. Tensors

a) Show that not all tensors in

$$\begin{aligned} V \otimes V &= \{T \mid T \text{ is a bilinear form over } V^* \times V^*\} \\ &= \{\text{linear combinations of tensors } v_1 \otimes v_2 \mid v_1, v_2 \in V\} \end{aligned}$$

are of the form $v_1 \otimes v_2$.

b) Identify $V \otimes W^*$ with the vector space $\mathcal{L}(W, V)$ of linear maps $W \to V$.

3. On the Lie derivative

In class the Lie derivative $L_X R$ of a tensor field R was defined in 'absolute terms', i.e. without reference to charts or to components. It was then shown that, for the case of R being of type $\binom{1}{1}$, the components of $L_X R$ are

$$(L_X R)^i{}_j = R^i{}_{j,k} X^k - R^k{}_j X^i{}_{,k} + R^i{}_k X^k{}_{,j}\,. \tag{1}$$

By contrast adopt here the point of view according to which R is simply given by its components ${R^i}_j$ together with the transformation law

$$ {\bar{R}^\alpha}_\beta = {R^i}_j \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^\beta} \tag{2} $$

under any change $x \mapsto \bar{x}$ of coordinates. Then take (1) as a definition of $L_X R$. Make sure it is well-defined by showing that ${(L_X R)^i}_j$ also obeys (2).

Hint: Find first the transformation law of ${R^i}_{j,k}$ (it is not that of a tensor)