# General relativity, exercise sheet 12.

*HS 08* *Due:* Fri, December 19, 2008

## 1. Tidal forces near a black hole

The tidal forces are the relative accelerations of nearby freely falling particles or, related to that, the stresses within an extended freely falling body. They are described by the equation of geodesic deviation

$$\nabla_u^2 n = R(u,n)u\,,$$

where $u$ is a time-like geodesic vector field and $n$ describes an infinitesimal relative displacement.

Show that the tidal forces remain finite as a particle falls through the Schwarzschild radius $r = 2m$. (The acceleration averaged over directions $n$ is not informative, since the Ricci tensor vanishes.) To this end let $\{e^\mu\}$ be the basis of 1-forms

$$e^0 = (1-2m/r)^{1/2}dt\,,\quad e^1 = (1-2m/r)^{-1/2}dr\,,\quad e^2 = rd\theta\,,\quad e^3 = r\sin\theta d\phi\,,$$

which has been chosen so that the metric reads

$$g = \eta_{\mu\nu}e^\mu e^\nu = e^0\otimes e^0 - \sum_{i=1}^{3} e^i\otimes e^i\,.$$

A computation (not to be performed) of the Riemann tensor $R(W,Z,X,Y) := (W, R(X,Y)Z)$ gives

$$\begin{aligned} R = \frac{m}{r^3}\big(&2(e^0\wedge e^1)\otimes(e^0\wedge e^1) - (e^0\wedge e^2)\otimes(e^0\wedge e^2) - (e^0\wedge e^3)\otimes(e^0\wedge e^3)\\ &+ (e^1\wedge e^2)\otimes(e^1\wedge e^2) + (e^1\wedge e^3)\otimes(e^1\wedge e^3) - 2(e^2\wedge e^3)\otimes(e^2\wedge e^3)\big)\,, \end{aligned}$$

where $\omega\wedge\widetilde{\omega} := \omega\otimes\widetilde{\omega} - \widetilde{\omega}\otimes\omega$ and hence $(e^\mu\wedge e^\nu)(X,Y) = X^\mu Y^\nu - X^\nu Y^\mu$ for $X = X^\mu e_\mu$.

Let $u$ be the 4-velocity of a particle in radial free fall.

i) In the attempt of finding the longitudinal (radial) tidal force one might compute

$$\frac{(n, R(u,n)u)}{(n,n)}$$

with $n = \partial/\partial r$. That diverges at $r = 2m$ and is wrong. Why?

ii) Take a vector $n$ with $(u,n) = 0$ and which is appropriate for longitudinal displacement. Show

$$\frac{(n, R(u,n)u)}{(n,n)} = \frac{2m}{r^3}\,,$$

which is finite at $r = 2m$.

*Hint:* $(u^0n^1 - u^1n^0)^2 - (u^0n^0 - u^1n^1)^2 = -((u^0)^2 - (u^1)^2)((n^0)^2 - (n^1)^2)\,.$

iii) Compute the transverse acceleration similarly.

**2. Surface gravity**

Compute the 4-acceleration $a = \nabla_u u$ of an observer staying at a fixed position $r, \theta, \phi$ in Schwarzschild coordinates (it has 4-velocity $u \parallel K$, the Killing vector). The time underlying $a$ is proper time $\tau$, since $\nabla_u = \nabla_{\partial/\partial\tau}$. Show that its length has a limit as $r \to 2m$, provided time $t$ is measured w.r.t. an observer at rest at infinity:

$$\kappa = \lim_{r \to 2m} -(a, a)^{1/2} \frac{d\tau}{dt} .$$

*Remark.* $\kappa$ is known as surface gravity and is related to the Hawking temperature. Indeed, that temperature depends on the relation of the above static observer to a freely falling one (i.e. on the acceleration of the former) as frequencies measured from infinity are concerned.