General relativity, exercise sheet 2.

HS 08 *Due:* Fri, October 10, 2008

1. Jacobi identity

i) Let X, Y, Z be vector fields on a manifold M. Verify that the commutator satisfies the Jacobi identity:

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 \, .$$

ii) Let $Y_1 \ldots Y_m$ be vector fields on an n-dimensional manifold M such that at each $p \in M$ they form a basis of the tangent space $T_p(M)$. Then, at each point, we may expand each commutator $[Y_\alpha, Y_\beta]$ in this basis, thereby defining the functions $C^\gamma{}_{\alpha\beta} = -C^\gamma{}_{\beta\alpha}$ by

$$[Y_\alpha, Y_\beta] = C^\gamma{}_{\alpha\beta} Y_\gamma \, . \tag{1}$$

Use the Jacobi identity to derive an equation satisfied by $C^\gamma{}_{\alpha\beta}$.

2. Lie groups and Lie brackets

Consider the group of regular, real $n \times n$ matrices:

$$\mathrm{GL}(n,\mathbb{R}) = \{m = (m_{ij})_{i,j=1}^n \mid m_{ij} \in \mathbb{R} \, , \det m \neq 0\}$$

equipped with matrix multiplication. The unit element is $e = (\delta_{ij})_{i,j=1}^n$. $\mathrm{GL}(n,\mathbb{R})$ is a differentiable manifold of dimension n^2.

The tangent space at e consists of tangents $\dot{m}(0)$ to curves $m(t)$ with $m(0) = e$ and is denoted by

$$\mathfrak{gl}(n,\mathbb{R}) = T_e(\mathrm{GL}(n,\mathbb{R})) = \{x = (x_{ij})_{i,j=1}^n \mid x_{ij} \in \mathbb{R}\} \, .$$

A (matrix) Lie group G is a subgroup and a submanifold of $\mathrm{GL}(n,\mathbb{R})$. Examples (besides of the trivial $G = \mathrm{GL}(n,\mathbb{R})$) are

a) the orthogonal group

$$\mathrm{O}(n) = \{r \in \mathrm{GL}(n,\mathbb{R}) \mid r^T r = e\} \, ,$$

where r^T is the transpose of r;

b) the Lorentz group

$$\mathrm{SO}(1,3) = \{l \in \mathrm{GL}(4,\mathbb{R}) \mid l^T \eta l = \eta\} \, ,$$

where $\eta = \mathrm{diag}(1,-1,-1,-1)$.

The tangent space at $e \in G$ consists of matrices:

$$\mathrm{Lie}(G) := T_e(G) \subset \mathfrak{gl}(n,\mathbb{R}) \, .$$

i) Find $\mathrm{Lie}(G)$ for G in the examples (a), (b).

ii) Show that for any $x_1, x_2 \in \mathrm{Lie}(G)$

$$\alpha_1 x_1 + \alpha_2 x_2 \in \mathrm{Lie}(G)\,, \quad (\alpha_1, \alpha_2 \in \mathbb{R})\,, \tag{2}$$
$$[x_1, x_2] := x_1 x_2 - x_2 x_1 \in \mathrm{Lie}(G)\,, \tag{3}$$

moreover,

$$[\alpha_1 x_1 + \alpha_2 x_2, x] = \alpha_1 [x_1, x] + \alpha_2 [x_2, x]\,. \tag{4}$$

Hint: If $m_1(t), m_2(t) \in G$ are curves through e, so are $m_i(\lambda_i t)$ $(i = 1, 2)$, $m_1(t)m_2(t)$ and, for any s, $m_1(t)m_2(s)m_1(t)^{-1}m_2(s)^{-1}$.

A linear space equipped with a bilinear, antisymmetric bracket $[\cdot, \cdot]$ satisfying the Jacobi identity is called a Lie algebra. $\mathrm{Lie}(G)$ is the Lie algebra of G.

For any $g \in G$, let λ_g be the left-multiplication on G:

$$\lambda_g : G \longrightarrow G \quad h \longmapsto gh\,.$$

It is a diffeomorphism. Among the vector fields X on G, consider those which are left-invariant

$$X = (\lambda_g)_* X\,.$$

Clearly, they form a linear space. Show that

iii) They also form a Lie algebra w.r.t. the Lie bracket.

iv) If the vector fields in (1) are left-invariant, then the functions $C^{\gamma}{}_{\alpha\beta}$ are constants (called structure constants).

v) The left-invariant vector fields are in bijective relation to the tangent vectors at e

$$X \longleftrightarrow x \in T_e(G) = \mathrm{Lie}(G)$$

such that $X_e = x$.

vi) The bijection is a Lie algebra isomorphism:

$$\alpha_1 X_1 + \alpha_2 X_2 \longleftrightarrow \alpha_1 x_1 + \alpha_2 x_2\,,$$
$$[X_1, X_2] \longleftrightarrow [x_1, x_2]\,.$$