General relativity, exercise sheet 3.

HS 08 *Due:* Fri, October 17, 2008

1. Parallel transport in polar coordinates

Consider the euclidean plane $\mathbb{R}^2 \ni (x^1, x^2) = x$ as a manifold with chart: $\mathrm{id} : x \mapsto x$. Define a cartesian parallel transport $T_x(\mathbb{R}^2) \ni v \mapsto v' \in T_{x'}(\mathbb{R}^2)$ along any curve by requiring that v and v' have the same components. Compute the Christoffel symbols of this parallel transport in polar coordinates r, φ.

2. Affine connections

Show: given an affine connection ∇ on the manifold M, $\tilde{\nabla}$ is an affine connection iff the difference $B(X,Y) := \nabla_X Y - \tilde{\nabla}_X Y$ has the property that

$$(\omega, X, Y) \longmapsto \langle \omega, B(W, Y) \rangle$$

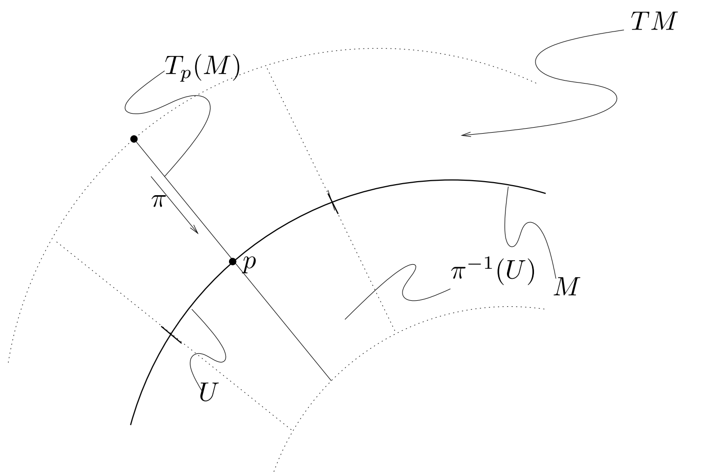
is a tensor field of type $\binom{1}{2}$ (i.e. the affine connections on M form an affine space over the vector space of these tensor fields).
What are the consequences of this fact for the Christoffel symbols?
Application: for every pair of affine connections $\nabla, \tilde{\nabla}$ the combination $(1-\alpha)\nabla + \alpha\tilde{\nabla}$ is also an affine connection.

3. Alternate view on parallel transport

The tangent bundle of a manifold M is the union of all its tangent spaces:

$$TM = \bigcup_{p \in M} T_p(M)$$

Let π be the projection $\pi : TM \to M$, $X \mapsto \pi(X) = p$ if $X \in T_p(M)$.



The tangent bundle becomes a differentiable manifold in its own right by means of charts defined as follows: If $K : U \to \mathbb{R}^n \;\; p \mapsto x = (x^1, \ldots x^n)$ is a patch for $U \subset M$, then

$$\begin{aligned} \tilde{K} : \pi^{-1}(U) &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \\ X &\longmapsto (x, \underline{X}), \end{aligned} \tag{1}$$

where $\underline{X} = (X^1, \ldots X^n)$ are the components of $X \in T_p(M)$ w.r.t. the coordinate basis, is a patch for $\pi^{-1}(U) := \bigcup_{p \in U} T_p(M) \subset TM$.

i) Let $\bar{x} \mapsto x$ be a coordinate change on $U \cap \bar{U} \subset M$. Compute the induced coordinate change on $\pi^{-1}(U \cap \bar{U})$. What is the matrix of its partial derivatives?

A parallel transport can be viewed as a map

$$\sigma_X : T_{\pi(X)}(M) \longrightarrow T_X(TM), \qquad Y \longmapsto \sigma_X(Y), \; (X \in TM)$$

such that in any chart (1)

$$\sigma_X(Y) \; ''='' \; \big(\underline{Y}, -\Gamma(\underline{Y}, \underline{X})\big)$$

with Γ linear in $\underline{Y}, \underline{X}$: $\Gamma(\underline{Y}, \underline{X})^i = \Gamma^i{}_{lk} Y^l X^k$ where $\Gamma^i{}_{lk}$ are the Christoffel symbols of the parallel transport ($''=''$ is short for the tangent map $\tilde{K}_*$; we refrain from giving a more intrinsic definition).

ii) Show that a vector $X(t)$ parallel transported along a curve $\gamma(t)$ in M is characterized by

$$\begin{aligned} \pi\big(X(t)\big) &= \gamma(t), \\ \dot{X}(t) &= \sigma_{X(t)}\big(\dot{\gamma}(t)\big). \end{aligned}$$

Hint: A family of vectors $X(t) \in T_{\gamma(t)}$ is a curve in TM. Hence $\dot{X}(t) \in T_{X(t)}(TM)$.

iii) Components of vectors in $T_X(TM)$ transform "tensorially" by means of the matrix found in i). Derive from that the non-tensorial transformation law for the Christoffel symbols.