General relativity, exercise sheet 5.

HS 08 *Due:* Fri, October 31, 2008

1. A free fall

A clock C_2 lies at fixed height h above a clock C_1 in a reference frame O in which the gravitational field g is constant. Two masses are dropped from C_2 in a short time interval Δt_2 as measured by the local clock. What is the time difference Δt_1 measured by C_1 between their arrival times?

Hint: Use the equivalence principle (hence make use of a local inertial frame) and special relativity.

2. The Newton's equation as a geodesic equation

Rewrite Newton's equation of motion $\ddot{\vec{x}} = -\vec{\nabla}\varphi$ in a gravitational potential $\varphi(\vec{x})$ as a geodesic equation for $(t, \vec{x}(t))$ in a 4-dimensional spacetime. Identify the Christoffel symbols of the affine connection. Show that the latter is (i) symmetric but (ii) cannot be metric.

Hint for (ii): Show that ${R^i}_{0k0} = \varphi_{,ik}$, ${R^i}_{jk0} = 0$, $(i, j, k \neq 0)$. Assuming the connection to be determined by a metric g, compute R_{i0j0} and R_{0ij0}, and obtain a contradiction to a symmetry of the Riemann tensor.

3. On the Riemann connection

Let $N \subset M$ be a submanifold of the manifold M with metric g. At any point $p \in N$ let $P_p : T_p(M) \to T_p(N)$ be the orthogonal projection associated to g_p, i.e.

$$g_p(X, Y) = g_p(P_p X, Y)\,, \tag{1}$$

where $X \in T_p(M)$ and $Y \in T_p(N)$. The metric g determines a Riemann connection $\nabla^{(M)}$ on M and, through its restriction, one on N too, $\nabla^{(N)}$. Show that

$$(\nabla^{(N)}_X Y)_p = P_p(\nabla^{(M)}_X Y)_p \tag{2}$$

for any vector fields X, Y on N.

Hint: Show that $P \circ \nabla^{(M)}$ has the properties of $\nabla^{(N)}$.