General relativity, exercise sheet 8.

HS 08 *Due:* Fri, November 21, 2008

1. Energy conditions

In the following a reference frame means a local basis e_α ($\alpha = 0, 1, 2, 3$) with $(e_\alpha, e_\beta) = \eta_{\alpha\beta}$. The 4-velocity of an observer at rest therein is $u = e_0$.

i) The 4-momentum $(p^\mu)_{\mu=0}^3 = (E/c, \vec{p})$ of a particle of mass $m \geq 0$ satisfies $E \geq 0$ in *every* reference frame. This is formulated in general covariant form as: p^μ is timelike or lightlike, and future oriented. The generalization on the energy-momentum tensor is: $T^{00} \geq 0$ in *every* reference frame. Formulate this **weak energy condition** in a covariant way.
Hint: The 4-velocity of an observer is timelike.

ii) The **strong energy condition** says $T^{00} + \sum_{i=1}^3 T^{ii} \geq 0$ in *every* reference frame. Formulate also this condition in a covariant way. Show: by use of the Einstein field equations this means that free falling matter attracts itself. More precisely: embed a reference geodesic $x(\tau)$, with initial 4-velocity $(dx/d\tau)|_{\tau=0} = e_0$, in a 1-parameter family of geodesics $x(\tau, \lambda)$, all with initial 4-velocity e_0. For the separation $n_a(\tau) = (dx/d\lambda)|_{\lambda=0}$ between close geodesics (see pg. 36) let the initial condition be $n_a(0) = e_a$, ($a = 1, 2$ or 3). Attraction means (averaging over the directions), that

$$-\sum_{a=1}^{3} (n_a, \nabla_{e_0}^2 n_a)\big|_{\tau=0} \leq 0 \, .$$

Remark: The strong energy condition does not imply the weak one.

iii) The **dominant energy condition** is a strenghtening of the weak energy condition. It requires the energy flow $(T^{\mu 0})_{\mu=0}^3$ to be timelike or lightlike, and future oriented ($T^{00} \geq 0$), in *every* reference frame. (That means that the propagation velocity of the energy is $\leq c$). Formulate this condition in a covariant way and show that it is equivalent to $T^{00} \geq |T^{\alpha\beta}|$ for every α, β in *every* reference frame (hence the name of the condition).

iv) What do the conditions (i-iii) imply for the ideal fluid, the electromagnetic field, and for the vacuum (cosmological term)?