Exercises for "Phenomenology of Particle Physics I"

Prof. Dr. A. Gehrmann
M. Ritzmann
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Exercise 18

Show the following distribution identity:

$$\int d^3x f_p(x)^* i \overleftrightarrow{\partial_0} f_q(x) \Big|_{t=0, p^0=E_{\vec{p}}, q^0=E_{\vec{q}}} = \delta^3(\vec{p}-\vec{q})$$

where we define $f(x)\overleftrightarrow{\partial_x} g(x) := -g(x)\partial_x f(x) + f(x)\partial_x g(x)$ and we have $f_p(x) = \frac{1}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2E_{\vec{p}}}}e^{-ipx}$.

Exercise 19

Starting from the fourier transform of the real Klein-Gordon field and the canonical momentum density conjugate to it (in the Heisenberg picture)

$$\phi(x) = \int d^3p \frac{1}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(f_p(x)a(\vec{p}) + f_p(x)^* a(\vec{p})^\dagger\right)\Big|_{p^0=E_{\vec{p}}}$$

$$\Pi(x) = \partial_0\phi(x) = \int d^3p \frac{1}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2E_{\vec{p}}}}(iE_{\vec{p}})\left(-f_p(x)a(\vec{p}) + f_p(x)^* a(\vec{p})^\dagger\right)\Big|_{p^0=E_{\vec{p}}}$$

with $E_{\vec{p}} = (|\vec{p}|^2 + m^2)^{1/2}$, derive the expressions for $a(\vec{p})$ and $a(\vec{p})^\dagger$ by inversion. The result is

$$a(\vec{p}) = \sqrt{(2\pi)^3}\sqrt{2E_{\vec{p}}}\int d^3x f_p(x)^* i\overleftrightarrow{\partial_0}\phi(x)$$

$$a(\vec{p})^\dagger = \sqrt{(2\pi)^3}\sqrt{2E_{\vec{p}}}\int d^3x \phi(x) i\overleftrightarrow{\partial_0} f_p(x).$$

– please turn over –

Exercise 20

Show that if we postulate the commutation relations

$$\left[a(\vec{p}), a(\vec{q})^\dagger\right] = \delta^3(\vec{p}-\vec{q})(2\pi)^3 2E_{\vec{p}}$$
$$\left[a(\vec{p}), a(\vec{q})\right] = \left[a(\vec{p})^\dagger, a(\vec{q})^\dagger\right] = 0$$

for a and $a^\dagger$ we arrive at the following commutation relations for the field and the canonical momentum density conjugate to it

$$\left[\phi(\vec{x},t), \Pi(\vec{x'},t)\right] = i\delta^3(\vec{x}-\vec{x'})$$
$$\left[\phi(\vec{x},t), \phi(\vec{x'},t)\right] = \left[\Pi(\vec{x},t), \Pi(\vec{x'},t)\right] = 0.$$

Exercise 21 (corrected)

We define

$$\Delta^\pm(x) = -\int_{C^\pm} \frac{dp^0}{2\pi} \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ipx}}{p^2-m^2}$$

where C^+ and C^- are contours in the complex p^0-plane, C^+ goes around $p^0 = E_{\vec{p}}$ once in counterclockwise direction, C^- goes around $p^0 = -E_{\vec{p}}$ once in counterclockwise direction.

- Use the residue theorem and the formula

  $$\int \frac{d^3p}{2E_{\vec{p}}} = \int d^4p\,\delta(p^2-m^2)\Theta(p^0)$$

  to show

  $$\Delta^\pm(x) = \mp i \int \frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)e^{\mp ipx}\Theta(p^0).$$

- Show that $[\phi(x), \phi(y)] = i\Delta(x-y) = i(\Delta^+(x-y) + \Delta^-(x-y))$ vanishes for spacelike $((x-y)^2 < 0)$ separation $x-y$.