Quantum Field Theory I, Exercise Set 1

HS 08 *Due:* 2/3 October 2008

1. Representations of SU(2) and particle physics

(i) Show that SU(2) is isomorphic to the 3-sphere S^3.
Hint: Show that every $A \in \mathrm{SU}(2)$ is of the form $A = a_0 \mathbb{1} + \sum_{i=1}^{3} \mathrm{i} a_i \sigma_i$, with $\sum_{i=0}^{3} a_i^2 = 1$, $a_i \in \mathbb{R}$. Here $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices.

(ii) The fundamental representation of SU(2) is given by $\rho : \mathrm{SU}(2) \to \mathrm{GL}(\mathbb{C}^2)$, $A \mapsto A$. The complex conjugate representation is given by $A \mapsto \overline{A}$. Show that these two representations are unitarely equivalent.
Hint: Cramer's rule for $A^{-1} = A^*$.

(iii) Show that the statement of (ii) is not true for SU(3).
Hint: Look at the character $\mathrm{Tr}\,\rho$ of a representation.

In the standard model of particle physics, hadronic matter consists of *quarks*, which are spin-1/2 particles. The lightest quarks, u and d, have nearly the same mass. Furthermore, the strong interaction seems not to depend on the flavour (u or d) of the quarks. This led Heisenberg and Wigner to the postulate that there is an SU(2)-symmetry between u and d quarks called isospin. By convention, u has isospin $1/2$ and d has isospin $-1/2$. Mesons are bound states of of a quark-antiquark pair. Baryons are bound states of three quarks.

(iv) Classify the mesons consisting of u, d, $\bar{u}$ and $\bar{d}$ quarks according to their total isospin.
Hint: Use the Clebsch-Gordan decomposition of $\mathcal{D}_{1/2} \otimes \mathcal{D}_{1/2}$. The singlet is the η meson, the triplet is formed by the π mesons.

(v) The Δ^{++} baryon is a bound state of three u-quarks with total spin 3/2. Its wave function is of the form

$$\psi_{\Delta^{++}} = \psi_{\text{spin}} \otimes \psi_{\text{flavour}} \otimes \psi_{\text{spatial}} \,. \tag{1}$$

Argue that the spin and flavour wavefunctions are symmetric. The orbital part of the wavefunction is also symmetric, since the minimisation of the total energy requires that the total angular momentum vanishes. Thus $\psi_{\Delta^{++}}$ is symmetric.

(vi) The spin-statistics theorem requires that spin-1/2 particles, such as quarks, be fermions, and consequently have an antisymmetric wavefunction. This contradiction was solved by introducing an additional quantum number, the *colour*.

Show that there must be at least three different colours.
Hint: The antisymmetric colour wavefunction ψ_{colour} can be identified with an antisymmetric tensor $T_{c_1 c_2 c_3}$, where c_1, c_2, c_3 label the colours of the three quarks.

(vii) Show that the colour gauge group SU(3) acts according to the trivial representation on ψ_{colour}. Thus Δ^{++} is "colourless" in agreement with the confinement hypothesis.

2. Direct integrals

In the representation theory of noncompact groups, one often needs to generalise the concept of direct sums of Hilbert spaces to direct integrals of Hilbert spaces, i.e. to replace expressions of the type $\bigoplus_{i\in I}\mathcal{H}_i$ with $\int_M^\oplus \mathrm{d}\rho(x)\,\mathcal{H}_x$.

The goal of this exercise is to outline the definition of a direct integral and consider some examples.

Let ρ be a measure on a set M. Let $\{\mathcal{H}_x\}_{x\in M}$ be a family of Hilbert spaces ("fibres"). Formally, the *direct integral* $\mathcal{H} = \int_M^\oplus \mathrm{d}\rho(x)\,\mathcal{H}_x$ is defined as the set of functions f on M satisfying $f(x) \in \mathcal{H}_x$ and

$$\int \mathrm{d}\rho(x)\,\|f(x)\|^2_{\mathcal{H}_x} \;<\; \infty\,.$$

(i) Show that $\mathcal{H}$ is a Hilbert space. In particular, give the scalar product $\langle\cdot\,,\cdot\rangle_{\mathcal{H}}$.

(ii) Show that the direct sum is a special case. More specifically, let I be a finite or countable index set and consider the direct sum $\mathcal{H} = \bigoplus_{i\in I}\mathcal{H}_i$. Show that $\mathcal{H}$ can be written as a direct integral.

(iii) For general (M,ρ) consider the case where the fibres are all identical: $\mathcal{H}_x = \mathbb{C}^n$ for all x. Show that

$$\int_M^\oplus \mathrm{d}\rho(x)\,\mathbb{C}^n \;=\; L^2(M,\mathrm{d}\rho)\otimes\mathbb{C}^n\,.$$

3. Representations of compact groups

(i) Let G be a compact group, $\mathcal{H}$ a Hilbert space, and U a unitary irreducible representation of G on $\mathcal{H}$ that is *strongly continuous*, i.e. the map $g\mapsto U(g)\psi$ is continuous for all $\psi\in\mathcal{H}$. Show that $\mathcal{H}$ is finite dimensional.
Hint: Use the facts that the image of a compact set under a continuous map is compact, and the unit sphere of a Hilbert space $\mathcal{H}$ if compact if and only if $\mathcal{H}$ is finite dimensional.

(ii) Show that it is enough to assume that U is *weakly continuous*, i.e. the map $g\mapsto \langle\psi\,,U(g)\phi\rangle$ is continous for all $\psi,\phi\in\mathcal{H}$.
Hint: Show that, in this case, weak continuity implies strong continuity (the converse is trivially true).

4. Lie groups and Lie algebras

Recall that a Lie group G is a smooth manifold that is also a group, such that the group operations are continuous. The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of G is by definition the tangent space of G at its identity $\mathbb{1}$. Elements $X\in\mathfrak{g}$ are called *generators* of G. Generators are conveniently specified by paths: $\gamma(\lambda)\in G$ satisfying $\gamma(0)=\mathbb{1}$ defines a generator X trough $X = \frac{\mathrm{d}}{\mathrm{d}\lambda}\gamma(\lambda)\big|_{\lambda=0}$. Let X_1 and X_2 be derivatives at 0 of the paths γ_1 and γ_2. Then the linear combination

$$\alpha_1X_1+\alpha_2X_2 \;=\; \frac{\mathrm{d}}{\mathrm{d}\lambda}\gamma_1(\alpha_1\lambda)\,\gamma_2(\alpha_2\lambda)\Big|_{\lambda=0}$$

and the Lie bracket

$$[X_1, X_2] = \frac{\partial}{\partial\lambda_1}\frac{\partial}{\partial\lambda_2}\gamma_1(\lambda_1)\gamma_2(\lambda_2)\gamma_1(\lambda_1)^{-1}\gamma_2(\lambda_2)^{-1}\Big|_{\lambda_1=\lambda_2=0}$$

are elements of $\mathfrak{g}$ and independent of the choice of paths. Moreover, $[\cdot,\cdot]$ is bilinear and antisymmetric, and satisfies the Jacobi identity.

(i) Suppose that $G \subset \mathrm{GL}(\mathbb{C}^n)$ is a matrix Lie group. Show that the Lie bracket defined above is equal to the commutator of matrices.

A *representation* L of a Lie algebra $\mathfrak{g}$ on $\mathbb{C}^n$ is a linear map from $\mathfrak{g}$ into the set of complex $n \times n$ matrices, such that

$$L([X_1, X_2]) = L(X_1)L(X_2) - L(X_2)L(X_1)\,.$$

(ii) Let G be a Lie group and $U : G \to \mathrm{GL}(\mathbb{C}^n)$ be a representation. Define the map $\dot{U}$ on $\mathfrak{g}$ through

$$\dot{U}(X) = \frac{\mathrm{d}}{\mathrm{d}\lambda}U(\gamma(\lambda))\Big|_{\lambda=0}, \qquad \text{for } X = \frac{\mathrm{d}}{\mathrm{d}\lambda}\gamma(\lambda)\Big|_{\lambda=0}.$$

Show that $\dot{U}$ is a representation of $\mathfrak{g}$ on $\mathbb{C}^n$. Inother words, to every representation of a Lie group corresponds a representation of its Lie algebra.

5. The Poincaré group

The Poincaré group, $\mathcal{P}_+^\uparrow = L_+^\uparrow \ltimes \mathbb{R}^4$, consists of elements (Λ, a) with multiplication

$$(\Lambda_1, a_1)\,(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \lambda_1 a_2)\,.$$

(i) Show that the Lie algebra $\mathfrak{g}$ of $\mathcal{P}_+^\uparrow$ is the linear space $\{(\varepsilon, \tau)\}$, where $\tau \in \mathbb{R}^4$ and $\varepsilon = \varepsilon^\mu{}_\nu$ satisfies $\varepsilon_{\mu\nu} + \varepsilon_{\nu\mu} = 0$. What is the dimension of $\mathfrak{g}$?

It is convenient to define a basis $\{J^{\mu\nu}, P_\sigma\}$ of $\mathfrak{g}$ through

$$\varepsilon = -\frac{\mathrm{i}}{2}\varepsilon_{\mu\nu}J^{\mu\nu}, \qquad \tau = -\mathrm{i}\tau^\sigma P_\sigma,,$$

where

$$J^{01} = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P_0 = \begin{pmatrix} \mathrm{i} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

etc.

(ii) In the lecture the vector $\vec{J}$ was defined through $J^{jk} = \varepsilon^{ijk}J_i$. Show that $J_i = -\frac{1}{2}\varepsilon_{ijk}J^{jk}$.

(iii) Compute the following Lie brackets: $[J^{\mu\nu},\, J^{\rho\sigma}]$, $[J^{\mu\nu},\, P_\sigma]$, $[P_\sigma,\, P_\rho]$.
Hint: Use the definition of $[\cdot, \cdot]$ from exercise 4. Alternatively, you can work in the representation described in (v) below.

(iv) The Pauli-Ljubanski vector is defined (in any representation) as $W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma$. Show that

$$(W_\mu) \;=\; \big(\vec{P}\cdot\vec{J}, P_0\vec{J} + \vec{P}\wedge\vec{K}\big)\,;$$

$\vec{J}$ and $\vec{K}$ were defined in class (lecture notes p. 49).

(v) Compute the commutators $[P_\mu,\, W_\sigma]$ and $[J^{\mu\nu},\, W_\sigma]$.
Hint: You will have to work in a representation. A convenient one is the space of functions $f : \mathbb{R}^4 \to \mathbb{C}$ with

$$((\Lambda, a)f)(x) \;=\; f(\Lambda^{-1}(x-a))\,.$$

Show first that, in this representation, $P_\mu \;=\; \mathrm{i}\partial_\mu$ and $J^{\mu\nu} \;=\; x^\mu P^\nu - x^\nu P^\mu$.