Quantum Field Theory I, Exercise Set 3.

HS 08 *Due:* 16/17 October 2008

1. Propagator of the Dirac Theory

Compute the Dirac propagator $\langle 0|\{\psi_a(x), \overline{\psi}_b(y)\}|0\rangle$ explicitly and show that it vanishes for spacelike $x - y$.

Hint: Work in the chiral representation of the Dirac algebra, show that

$$\begin{pmatrix} \hat{P} & m \\ m & P \end{pmatrix} \gamma^0 = (\gamma^\mu \partial_\mu + m) \,. \tag{1}$$

2. Non Relativistic Limit of the Dirac Theory and Landau Levels

In the lectures the non-relativistic limit of the Dirac equation with an electromagnetic field was discussed.

(i) Show that equations (45) and (46) in the appendix to Chapter 4 are identical, i.e.,

$$H^0 = \frac{(\vec{\sigma} \cdot \vec{\pi})^2}{2m} + e\phi = \frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m} + e\phi - \frac{e\hbar}{2mc}\vec{\sigma} \cdot \vec{B} \,. \tag{2}$$

The last term on the right side of the above equation should be compared with the Zeeman term in the Pauli Hamiltonian of a non-relativistic electron, i.e.,

$$-g\frac{e}{2mc}\vec{S} \cdot \vec{B} \,, \tag{3}$$

where g is the gyromagnetic factor and $\vec{S} = \frac{\hbar}{2}\sigma$. The Dirac theory thus implies $g = 2$.

(ii) We choose $\vec{B} = B\vec{e}_3$ and $\phi = 0$. Then the Hamiltonian (2) reduces to

$$H = \frac{1}{2m}(\pi_x^2 + \pi_y^2 + \pi_z^2) - \frac{e\hbar B}{2mc}\sigma_3 \,. \tag{4}$$

Show that the spectrum of this Hamiltonian is given by

$$E_{n,\,s,\,k} = \hbar\omega_c(n + \frac{1}{2}) + \hbar\omega_c\, s + \frac{(\hbar k)^2}{2m} \,, \tag{5}$$

where $n = 0, 1, 2 \ldots$, $s = \pm\frac{1}{2}$, $k \in \mathbb{R}$ and $\omega_c = \frac{|eB|}{mc}$ is the cyclotron frequency. This spectrum is named after Lev Landau.

Hint: Use that $\vec{A} = \frac{1}{2}\vec{B} \wedge \vec{x}$. *Show that the* π_i *satisfy the Heisenberg commutation relations, i.e.,*

$$[\pi_x,\, \pi_y] = i\frac{eB}{c}\hbar\mathbb{1} \,, \qquad [\pi_x,\, \pi_z] = [\pi_y,\, \pi_z] = 0 \,. \tag{6}$$

It follows that $[H,\, \pi_z] = 0$. *Define then the following operators*

$$a := \sqrt{\frac{cm}{2eB}}(\pi_x + \mathrm{i}\pi_y) \,, \quad a^* = \sqrt{\frac{cm}{2eB}}(\pi_x - \mathrm{i}\pi_y) \tag{7}$$

and rewrite the Hamiltonian using these operators and their commutation relations.

(iii)* Show that the eigenvalues $E_{n,s,k}$ have an infinite degeneracy.

3. Group action on a manifold

In this exercise we study some examples of a group acting on a manifold. A group G is said to *act on a set* X (from the left) if there is a mapping $G \times X \to X$, $(g, x) \mapsto g \cdot x$ that satisfies

$$g \cdot (h \cdot x) = (gh) \cdot x \,, \qquad e \cdot x = x \,,$$

for all $g, h \in G$ and $x \in X$. The *orbit* of a point $x \in X$ is defined as the set

$$G \cdot x = \{g \cdot x : g \in G\} \,.$$

(i) Consider the proper, orthochronous Lorentz group $L_+^\uparrow$ acting on Minkowski space $\mathbb{M}^4$ through

$$p \mapsto \Lambda p \,.$$

This clearly defines a group action. Determine all of its orbits.

(ii) Consider the two-dimensional torus $\mathbb{T}^2$, given by

$$\mathbb{R}^2 / \sim \,,$$

where $x \sim y$ means $x - y \in \mathbb{Z}^2$. Let $a, b \in \mathbb{R}$ and define the action of the additive group $\mathbb{R}$ on $\mathbb{T}^2$ through

$$t \cdot x = x + (a, b)t \,.$$

Determine its orbits.
Hint: Consider the two cases $a/b \in \mathbb{Q}$ and $a/b \notin \mathbb{Q}$.

(iii) A projective unitary representation is a group action on the set of rays $\mathcal{H}/\sim$ of a Hilbert space $\mathcal{H}$. Here $\Psi \sim \Phi$ means $\Psi = \mathrm{e}^{\mathrm{i}\alpha}\Phi$ for some $\alpha \in \mathbb{R}$. Consider the additive group $\mathbb{R}^2$ projectively represented on $\mathcal{H}$:

$$T_a T_b = \mathrm{e}^{\mathrm{i}\varphi(a,b)}\, T_{a+b} \,.$$

Without loss of generality (why?), we assume that $T_0 = \mathbb{1}$.

(a) Show that

$$T_a T_b = \mathrm{e}^{\mathrm{i}\psi(a,b)}\, T_b T_a \,,$$

where $\psi(a, b)$ is antisymmetric.

(b) Assume that ψ is bilinear. Show that

$$T_a T_b = \mathrm{e}^{-\mathrm{i}c\,(a_1 b_2 - a_2 b_1)}\, T_b T_a$$

for some $c \in \mathbb{R}$. These are the *Weyl relations.*

(c) Write the unitary operator T_a using the self-adjoint generators X and Y:

$$T_a = \mathrm{e}^{-\mathrm{i}(a_1 X + a_2 Y)} \,.$$

Show that the Weyl relations imply the *Heisenberg commutation relations*

$$[X, Y] = \mathrm{i}c \,.$$