Quantum Field Theory I, Exercise Set 4.

HS 08 *Due:* 23/24 October 2008

1. Scalar Field Theory

Consider a complex scalar field with Lagrangian density

$$\mathcal{L} = \partial_\mu \overline{\varphi} \partial^\mu \varphi - m^2 \varphi \overline{\varphi} - \frac{g}{2} (\varphi \overline{\varphi})^2 \,. \tag{1}$$

(i) Find the Euler-Lagrange equations of motion for $\varphi(t,\,\vec{x})$, $\overline{\varphi}(t,\,\vec{x})$, respectively.

(ii) Show that the Hamiltonian density function $\mathscr{H} = \mathscr{H}(\varphi,\,\overline{\varphi},\,\nabla\varphi,\,\nabla\overline{\varphi},\,\pi,\,\overline{\pi})$, where $\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)}$, is given by

$$\mathscr{H} = \pi\overline{\pi} + \nabla\varphi \cdot \nabla\overline{\varphi} + m^2 \varphi\overline{\varphi} + \frac{g}{2}(\varphi\overline{\varphi})^2 \,. \tag{2}$$

(iii) The Poisson bracket for two functions $F = F(\varphi, \overline{\varphi},\,\pi,\,\overline{\pi})$ and $G = G(\varphi, \overline{\varphi},\,\pi,\,\overline{\pi})$ on phase space is defined as

$$\begin{aligned} \{F,\,G\} := & \int \mathrm{d}^3x \left(\frac{\delta F}{\delta\pi(\vec{x})} \frac{\delta G}{\delta\varphi(\vec{x})} - \frac{\delta F}{\delta\varphi(\vec{x})} \frac{\delta G}{\delta\pi(\vec{x})} \right) \\ & + \int \mathrm{d}^3x \left(\frac{\delta F}{\delta\overline{\pi}(\vec{x})} \frac{\delta G}{\delta\overline{\varphi}(\vec{x})} - \frac{\delta F}{\delta\overline{\varphi}(\vec{x})} \frac{\delta G}{\delta\overline{\pi}(\vec{x})} \right) , \end{aligned}$$

where $\varphi(\vec{x}) \equiv \varphi(0, \vec{x})$ etc. The Poisson bracket is bilinear and antisymmetric. It has the derivation property, i.e., $\{FG,\,H\} = F\{G,\,H\} + \{F,\,H\}\,G$, and satisfies the Jacobi identity. Show that

$$\begin{aligned} &\{\pi(\vec{x}), \varphi(\vec{y})\} = \delta(\vec{x}-\vec{y}) \,, && \{\pi(\vec{x}),\,\pi(\vec{y})\} = \{\varphi(\vec{x})\,, \varphi(\vec{y})\} = 0 \,, \\ &\{\overline{\pi}(\vec{x}), \overline{\varphi}(\vec{y})\} = \delta(\vec{x}-\vec{y}) \,, && \{\overline{\pi}(\vec{x}),\,\overline{\pi}(\vec{y})\} = \{\overline{\varphi}(\vec{x})\,, \overline{\varphi}(\vec{y})\} = 0 \,, \\ &\{\overline{\pi}(\vec{x}), \varphi(\vec{y})\} = 0 \,, && \{\pi(\vec{x}),\,\overline{\varphi}(\vec{y})\} = 0 \,. \end{aligned} \tag{3}$$

(iv) The Hamiltonian equations of motion are given by

$$\partial_t \varphi_t(\vec{x}) = \{H,\,\varphi_t(\vec{x})\} \quad \text{and} \quad \partial_t \pi_t(\vec{x}) = \{H,\,\pi_t(\vec{x})\} \,,$$

with $\varphi_0(\vec{x}) = \varphi(0, \vec{x})$, $\varphi_0(\vec{x}) = \varphi(0, \vec{x})$, and $H = \int \mathrm{d}^3x \mathscr{H}$. Show that these equations and their complex conjugates are the same as those found in (i).

Hint: Use that the energy is conserved, i.e., $H[\varphi_t,\,\overline{\varphi}_t,\,\pi_t,\,\overline{\pi}_t] = H[\varphi,\,\overline{\varphi},\,\pi,\,\overline{\pi}]$.

(v) We set $g = 0$ in the Lagrangian density (1). We let $\hat{\varphi}(\vec{x})$ and $\hat{\pi}(\vec{x})$ denote the quantum field and its conjugate momentum, respectively, introduced in Chapter 3. The classical theory is quantized by replacing the classical fields and the Poisson bracket by these quantum fields and the commutator $\frac{\mathrm{i}}{\hbar}[\cdot\,,\,\cdot]$, respectively. We demand that (3) hold with the replaced objects. The Heisenberg equations of motion are given by

$$\partial_t \hat{\varphi}_t(\vec{x}) = \frac{\mathrm{i}}{\hbar}[\hat{H},\,\hat{\varphi}_t(\vec{x})] \quad \text{and} \quad \partial_t \hat{\pi}_t(\vec{x}) = \frac{\mathrm{i}}{\hbar}[\hat{H},\,\hat{\pi}_t(\vec{x})] \,,$$

and their adjoint versions. Show that these equations, for the quantized Hamiltonian (2), are equivalent to the Klein-Gordon equations for the quantum fields.

2. Reading Excercise: Symmetries in Classical Mechanics

(i) Read pages 120-135 in the lecture notes.

(ii) Given a one-parameter family of trajectories $q^i_\lambda(\cdot)$, $i = 1, 2, \ldots, N$, define what you mean by $\delta S = 0$.

(iii)* Starting from equation (5.29) in the lecture notes, derive the conserved quantity for the symmetry Φ_λ.

3. Noether Current for U(1) Symmetry

The Langrangian density (1) has an inner U(1) symmetry: it is invariant under the action of U(1) given by

$$\varphi(x) \mapsto \mathrm{e}^{-\mathrm{i}\lambda}\varphi(x)\,, \quad \overline{\varphi}(x) \mapsto \mathrm{e}^{\mathrm{i}\lambda}\overline{\varphi}(x)\,, \quad x_\lambda \equiv x\,.$$

Compute the corresponding Noether current.

4. Energy-Momentum Tensor

In the theory of general relativity the metric on space-time is a dynamical entity, i.e., $g_{\mu\nu} = g_{\mu\nu}(x)$, $x \in \mathbb{M}^4$. The Lagrangian density of a scalar field in this setting is given by

$$\mathcal{L}(x) = \frac{1}{2}(\partial_\mu\varphi)g^{\mu\nu}(x)(\partial_\nu\varphi) - V(\varphi)\,,$$

where V is some potential, and $g^{\mu\nu}(x)$ is the inverse of $g_{\mu\nu}(x)$. The action is given by

$$S = \int \mathrm{d}^4x\,\sqrt{-\mathrm{det}g}\,\mathcal{L}(x)\,.$$

Show that

$$T^{\mu\nu}(x) = -2\frac{\delta S}{\delta g_{\mu\nu}(x)}\bigg|_{g=\eta}\,,$$

where η is the standard Minkowski metric and $T^{\mu\nu}$ is the energy-momentum tensor defined by

$$T^{\mu\nu}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi(x))}(\partial^\nu\varphi)(x) - \mathcal{L}(x)\eta^{\mu\nu}\,.$$

Hint: Show that $\delta\mathrm{det}g = (\mathrm{det}g)\,g^{\mu\nu}\delta g_{\mu\nu}$ *and* $\delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}$.