Quantum Field Theory I, Exercise Set 5

HS 08 *Due:* 30/31 October 2008

1. Gupta-Bleuler formalism and physical Hilbert space

Complete the proof of the Lemma on page 164 in the lecture notes.

Hint: Convince yourself that any state in the (unphysical) Fock space $\mathcal{F}$ can be written as a linear combination of vectors of the form

$$\psi = \prod_{j=1}^{m} a^*_{r_j}(f_j) \prod_{i=1}^{n} (a^*_3(g_i) + a^*_0(h_i)) \, |0\rangle \, ,$$

where $r_j = 1$ or 2, and f_i, g_i, h_i are test functions on $\mathbb{R}^3$. The vector ψ is in the physical Fock space $\mathcal{F}_{\text{phys}}$ if and only if

$$[a_3(\mathbf{k}) - a_0(\mathbf{k})]\psi = 0 \, , \quad \forall \, \mathbf{k} \, .$$

Proceeding by induction on n, show that this implies $g_i = -h_i$, $i \leq n$.

2. Hamiltonian formulation of the EM field in the Coulomb gauge

In class the electromagnetic field was quantised in the Lorenz gauge (Gupta-Bleuler). The goal of this exercise is to work through the quantisation of the electromagnetic field in the Coulomb gauge.

(i) A vector field $\mathbf{X}$ on $\mathbb{R}^3$ may be decomposed into its transverse and longditudinal parts: $\mathbf{X} = \mathbf{X}_{\text{T}} + \mathbf{X}_{\text{L}}$, where $\nabla \cdot \mathbf{X}_{\text{T}} = 0$ and $\nabla \wedge \mathbf{X}_{\text{L}} = 0$. Find explicit expressions for $\mathbf{X}_{\text{T}}$ and $\mathbf{X}_{\text{L}}$ and show that

$$(X_{\text{T}})_i(\mathbf{x}) \; = \; \sum_j \int \mathrm{d}\mathbf{y} \; \delta^T_{ij}(\mathbf{x} - \mathbf{y}) X_j(\mathbf{y}) \, ,$$

where δ^T is the *transverse delta function*

$$\delta^T_{ij}(\mathbf{x} - \mathbf{y}) \; := \; \left(\delta_{ij} - \partial_i \partial_j \Delta^{-1}\right) \delta(\mathbf{x} - \mathbf{y}) \, ,$$

and the operator Δ^{-1} is defined by

$$(\Delta^{-1} f)(\mathbf{x}) \; := \; \frac{1}{4\pi} \int \mathrm{d}\mathbf{y} \; \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \, .$$

Hint: Use the identity $\Delta \mathbf{X} = \nabla(\nabla \cdot \mathbf{X}) - \nabla \wedge (\nabla \wedge \mathbf{X})$.

(ii) Introduce the scalar and vector potentials ϕ and $\mathbf{A}$, which satisfy $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$ and $\mathbf{B} = \nabla \wedge \mathbf{A}$. Show that, in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the Maxwell equations read

$$-\Delta\phi \; = \; \rho \, , \qquad \Box \mathbf{A} \; = \; \mathbf{j} - \partial_t \nabla \phi \, .$$

Hence ϕ is determined by $\phi = -\Delta^{-1}\rho$. All that remains is a wave equation for $\mathbf{A}$, whose solution is uniquely determined by $\mathbf{A}$ and $\partial_t \mathbf{A}$ at $t = 0$.

(iii) Let us first consider the free electromagnetic field, $\rho = 0$ and $\mathbf{j} = 0$. The phase space of the electromagnetic field is given by

$$\Gamma_{\text{EM}} \; := \; \{(\mathbf{A}, \mathbf{E}) \, : \, \nabla \cdot \mathbf{A} \; = \; \nabla \cdot \mathbf{E} \; = \; 0\} \, .$$

We introduce a Poisson bracket $\{\cdot, \cdot\}$ on Γ through

$$\{A_i(\mathbf{x}), E_j(\mathbf{y})\} \; = \; \delta^T_{ij}(\mathbf{x} - \mathbf{y}) \, , \tag{1}$$

(all other brackets vanish). Imposing the usual properties of $\{\cdot, \cdot\}$ – bilinearity, Jacobi identity and the Leibniz rule in both arguments – determines $\{\cdot, \cdot\}$ uniquely. Show that

$$\left\{ \int \mathrm{d}\mathbf{x} \; \mathbf{u}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x}) \, , \int \mathrm{d}\mathbf{x} \; \mathbf{v}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) \right\} \; = \; \int \mathrm{d}\mathbf{x} \; \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, , \tag{2}$$

if $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} = 0$.

(iv) The Hamilton function is defined by

$$H = \frac{1}{2} \int \mathrm{d}\mathbf{x} \, \big(\mathbf{E}^2(\mathbf{x}) + \mathbf{B}^2(\mathbf{x})\big) \,. \tag{3}$$

Show that the Hamiltonian equations of motion are equivalent to the Maxwell equations.

(v) In order to quantise the electromagnetic field, it is more convenient to work in momentum space:

$$\mathbf{A}(\mathbf{x}) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}} \, \mathbf{q}(\mathbf{k}) \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \,, \qquad \mathbf{E}(\mathbf{x}) = -\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}} \, \mathbf{p}(\mathbf{k}) \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \,.$$

Show that the conditions $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{E} = 0$ and $\mathbf{A}, \mathbf{E}$ real imply that

$$\mathbf{A}(\mathbf{x}) = \sum_{\lambda=1,2} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2|\mathbf{k}|}} \Big(\boldsymbol{\varepsilon}_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + \overline{\boldsymbol{\varepsilon}}_\lambda(\mathbf{k}) \overline{a}_\lambda(\mathbf{k}) \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \Big) \,,$$

$$\mathbf{E}(\mathbf{x}) = \mathrm{i} \sum_{\lambda=1,2} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}} \frac{\sqrt{|\mathbf{k}|}}{\sqrt{2}} \Big(\boldsymbol{\varepsilon}_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} - \overline{\boldsymbol{\varepsilon}}_\lambda(\mathbf{k}) \overline{a}_\lambda(\mathbf{k}) \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \Big) \,,$$

where $\boldsymbol{\varepsilon}_1(\mathbf{k})$ and $\boldsymbol{\varepsilon}_2(\mathbf{k})$ are orthonormal complex vectors, both orthogonal to $\mathbf{k}$, and $a_\lambda(\mathbf{k})$ is a complex function.

(vi) Show that the Hamilton function (3) in the new coordinates $a_\lambda(\mathbf{k}), \overline{a}_\lambda(\mathbf{k})$ is given by

$$H = \sum_{\lambda=1,2} \int \mathrm{d}\mathbf{k} \, |\mathbf{k}| \, |a_\lambda(\mathbf{k})|^2 \,.$$

(vii) Show that the Poisson bracket is given by

$$\{a_\lambda(\mathbf{k}) \,,\, \overline{a}_{\lambda'}(\mathbf{k}')\} = \mathrm{i} \, \delta_{\lambda\lambda'} \, \delta(\mathbf{k} - \mathbf{k}') \,,$$

(all other brackets vanish). Compute the Hamiltonian equations of motion for $a_\lambda(\mathbf{k}), \overline{a}_\lambda(\mathbf{k})$.

(viii) Quantise the free electromagnetic field as follows. Replace $a_\lambda(\mathbf{k}) \to \widehat{a}_\lambda(\mathbf{k})$ and $\overline{a}_\lambda(\mathbf{k}) \to \widehat{a}^*_\lambda(\mathbf{k})$ in the classical expressions and write creation operators to the left of annihilation operators in products. Here $\widehat{a}^*_\lambda(\mathbf{k})$ and $\widehat{a}_\lambda(\mathbf{k})$ are bosonic creation and annihilation operators satisfying

$$\big[\widehat{a}_\lambda(\mathbf{k}), \widehat{a}^*_{\lambda'}(\mathbf{k}')\big] = \delta_{\lambda\lambda'} \, \delta(\mathbf{k} - \mathbf{k}') \,.$$

Calculate $\widehat{\mathbf{A}}(t, \mathbf{x})$, defined as the solution of the Heisenberg equation of motion

$$\mathrm{i}\partial_t \widehat{\mathbf{A}}(t, \mathbf{x}) = \big[\widehat{H}, \widehat{\mathbf{A}}(t, \mathbf{x})\big] \,.$$

(ix) Calculate

$$\big\langle 0 \,\big|\, \widehat{A}_i(t, \mathbf{x}) \widehat{A}_j(s, \mathbf{y}) \,\big|\, 0 \big\rangle \,.$$

(x)* Let us now introduce N charged particles with masses m_i, charges e_i, positions $\mathbf{x}_i$ and momenta $\mathbf{p}_i$, for $i = 1, \dots, N$. The phase space is $\Gamma = \Gamma_{\mathrm{EM}} \times \mathbb{R}^{6N}$. From now on we call the divergence-free field $\mathbf{E}$ of part (iii) $\mathbf{E}_{\mathrm{T}}$. The Poisson bracket is defined by (1) for $\mathbf{A}$ and $\mathbf{E}_{\mathrm{T}}$, as well as

$$\{p_{ia}, x_{jb}\} = \delta_{ij} \delta_{ab} \,.$$

All other brackets vanish. The Hamilton function is given by

$$H = \sum_{i=1}^N \frac{1}{2m_i} \big(\mathbf{p}_i - e_i \mathbf{A}(\mathbf{x}_i)\big)^2 + \sum_{1 \le i < j \le N} \frac{e_i e_j}{4\pi |\mathbf{x}_i - \mathbf{x}_j|} + \frac{1}{2} \int \mathrm{d}\mathbf{x} \, \big(\mathbf{E}_{\mathrm{T}}^2(\mathbf{x}) + \mathbf{B}^2(\mathbf{x})\big) \,.$$

Show that the Hamiltonian equations of motion are equivalent to Maxwell's equations (in the Coulomb gauge) coupled with Newton's equations, where:

$$\rho(\mathbf{x}) = \sum_{i=1}^N e_i \, \delta(\mathbf{x} - \mathbf{x}_i) \,, \qquad \mathbf{j}(\mathbf{x}) = \sum_{i=1}^N e_i \, \dot{\mathbf{x}}_i \, \delta(\mathbf{x} - \mathbf{x}_i) \,,$$

as well as

$$\mathbf{B} = \nabla \wedge \mathbf{A} \,, \qquad \mathbf{E} = \mathbf{E}_{\mathrm{T}} + \mathbf{E}_{\mathrm{L}} \,,$$

and

$$\mathbf{E}_{\mathrm{L}} = -\nabla \phi \,, \qquad \phi = -\Delta^{-1} \rho \,.$$