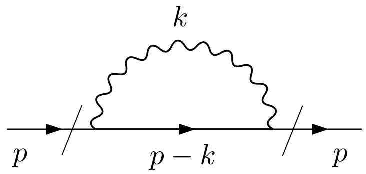
Quantum Field Theory I, Exercise Set 12.

HS 08 *Due:* 18/19 December 2008

1. Determination of Z_1 and δm to one-loop order

The electron self-energy is given, to one-loop order, by

$-\mathrm{i}\Sigma^{(2)}(p) =$  .

This diagram is logarithmically divergent. In the following we regularise it using dimensional regularisation.

(i) Write down the Feynman amplitude of the above diagram in D space-time dimensions. Using the Feynman parametrisation show that

$$-\mathrm{i}\Sigma^{(2)}(p) = -e_{\mathrm{R}}^2 \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^D\tilde{k}}{(2\pi)^D} \frac{(2-D)(1-z)\not{p} + D m_{\mathrm{R}}}{(\tilde{k}^2 + s + \mathrm{i}0)^2},$$

where $\tilde{k} = k - zp$ and $s = z(1-z)p^2 - \mu^2(1-z) - m_{\mathrm{R}}^2 z$.

(ii) Show that evaluating the momentum integral for small $\varepsilon = 4 - D$ yields

$$-\mathrm{i}\Sigma^{(2)}(p) = \frac{-\mathrm{i}e_{\mathrm{R}}^2}{8\pi^2}\left(\frac{4\pi}{m_{\mathrm{R}}^2}\right)^{\frac{\varepsilon}{2}} \frac{\Gamma(1+\frac{\varepsilon}{2})}{\varepsilon} \int_0^1 \mathrm{d}z \frac{(\varepsilon-2)(1-z)\not{p} + (4-\varepsilon)m_{\mathrm{R}}}{(z(1-z)\frac{p^2}{m_{\mathrm{R}}^2} - (1-z)\frac{\mu^2}{m_{\mathrm{R}}^2} - z)^{\frac{\varepsilon}{2}}}. \qquad (1)$$

Note that one can explicitely do the integral over z. But for our purposes the above expression is sufficient. It may be rewritten as

$$-\mathrm{i}\Sigma^{(2)}(p) = \mathrm{i}e_{\mathrm{R}}^2 A(p^2)(\not{p} - m_{\mathrm{R}}) + \mathrm{i}e_{\mathrm{R}}^2 B(p^2) m_{\mathrm{R}}.$$

Determine the functions A and B.

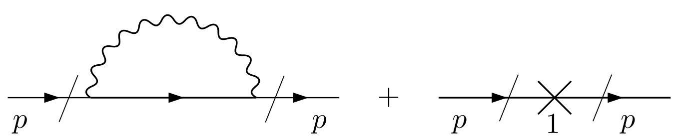
Hint: Use that $\Gamma(x+1) = x\Gamma(x)$.

(iii) Extract the diverging terms of (1) by showing that

$$-\mathrm{i}\Sigma^{(2)}(p) = \frac{\mathrm{i}e_{\mathrm{R}}^2}{8\pi^2\varepsilon}\Gamma(1+\frac{\varepsilon}{2})(\not{p} - m_{\mathrm{R}}) - \frac{3\mathrm{i}e_{\mathrm{R}}^2}{8\pi^2\varepsilon}\Gamma(1+\frac{\varepsilon}{2})m_{\mathrm{R}} + \text{ convergent terms}.$$

Hint: For small ε one has $s^{\frac{\varepsilon}{2}} = 1 - \frac{\varepsilon}{2}\ln s + O(\varepsilon^2)$.

(iv) The renormalised electron self-energy is given diagramatically to second order by

$-\mathrm{i}\Sigma_{\mathrm{R}}^{(2)}(p) =$  .

Recalling that $Z_1 = 1+\sum_{n=1}^{\infty} \xi_{1,n} e_{\mathrm{R}}^{2n}$ and $\delta m := m - m_{\mathrm{R}} = \sum_{n=1}^{\infty} \mu_n e_{\mathrm{R}}^{2n}$, convience yourself that the amplitude of the second diagram is given by $\mathrm{i}e_{\mathrm{R}}^2 \xi_{1,1}(\not{p} - m_{\mathrm{R}}) - \mathrm{i}e_{\mathrm{R}}^2 \mu_1$.

We impose the renormalisation conditions

$$\Sigma_R^{(2)}(p)\Big|_{\not{p}=m_{\mathrm{R}}} = 0 \quad \text{and} \quad \frac{\partial}{\partial \not{p}} \Sigma_R^{(2)}(p)\Big|_{\not{p}=m_{\mathrm{R}}} = 0\,.$$

Determine $\xi_{1,1}$ and μ_1 in terms of $A(p^2)$ and $B(p^2)$ and their derivatives.