

Quantum Field Theory I, Exercise Set 12.

HS 08

Due: 18/19 December 2008

1. Determination of Z_1 and δm to one-loop order

The electron self-energy is given, to one-loop order, by

$$-i\Sigma^{(2)}(p) = \text{Diagram: A fermion line with momentum } p \text{ enters from the left, a wavy photon line with momentum } k \text{ loops back to the fermion line, and the fermion line exits with momentum } p \text{ on the right. The internal fermion line has momentum } p-k \text{ and the photon line has momentum } k \text{ above it.}$$

This diagram is logarithmically divergent. In the following we regularise it using dimensional regularisation.

- (i) Write down the Feynman amplitude of the above diagram in D space-time dimensions. Using the Feynman parametrisation show that

$$-i\Sigma^{(2)}(p) = -e_R^2 \int_0^1 dz \int \frac{d^D \tilde{k}}{(2\pi)^D} \frac{(2-D)(1-z)\not{p} + Dm_R}{(\tilde{k}^2 + s + i0)^2},$$

where $\tilde{k} = k - zp$ and $s = z(1-z)p^2 - \mu^2(1-z) - m_R^2 z$.

- (ii) Show that evaluating the momentum integral for small $\varepsilon = 4 - D$ yields

$$-i\Sigma^{(2)}(p) = \frac{-ie_R^2}{8\pi^2} \left(\frac{4\pi}{m_R^2} \right)^{\frac{\varepsilon}{2}} \frac{\Gamma(1 + \frac{\varepsilon}{2})}{\varepsilon} \int_0^1 dz \frac{(\varepsilon - 2)(1-z)\not{p} + (4 - \varepsilon)m_R}{(z(1-z)\frac{p^2}{m_R^2} - (1-z)\frac{\mu^2}{m_R^2} - z)^{\frac{\varepsilon}{2}}}. \quad (1)$$

Note that one can explicitly do the integral over z . But for our purposes the above expression is sufficient. It may be rewritten as

$$-i\Sigma^{(2)}(p) = ie_R^2 A(p^2)(\not{p} - m_R) + ie_R^2 B(p^2)m_R.$$

Determine the functions A and B .

Hint: Use that $\Gamma(x+1) = x\Gamma(x)$.

- (iii) Extract the diverging terms of (1) by showing that

$$-i\Sigma^{(2)}(p) = \frac{ie_R^2}{8\pi^2\varepsilon} \Gamma(1 + \frac{\varepsilon}{2})(\not{p} - m_R) - \frac{3ie_R^2}{8\pi^2\varepsilon} \Gamma(1 + \frac{\varepsilon}{2})m_R + \text{convergent terms}.$$

Hint: For small ε one has $s^{\frac{\varepsilon}{2}} = 1 - \frac{\varepsilon}{2} \ln s + O(\varepsilon^2)$.

- (iv) The renormalised electron self-energy is given diagrammatically to second order by

$$-i\Sigma_R^{(2)}(p) = \text{Diagram: One-loop self-energy diagram} + \text{Diagram: A fermion line with momentum } p \text{ and a cross symbol labeled } 1 \text{ on the line, representing a counterterm.}$$

Recalling that $Z_1 = 1 + \sum_{n=1}^{\infty} \xi_{1,n} e_{\mathbb{R}}^{2n}$ and $\delta m := m - m_{\mathbb{R}} = \sum_{n=1}^{\infty} \mu_n e_{\mathbb{R}}^{2n}$, convince yourself that the amplitude of the second diagram is given by $ie_{\mathbb{R}}^2 \xi_{1,1} (\not{p} - m_{\mathbb{R}}) - ie_{\mathbb{R}}^2 \mu_1$.

We impose the renormalisation conditions

$$\Sigma_{\mathbb{R}}^{(2)}(p) \Big|_{\not{p}=m_{\mathbb{R}}} = 0 \quad \text{and} \quad \frac{\partial}{\partial \not{p}} \Sigma_{\mathbb{R}}^{(2)}(p) \Big|_{\not{p}=m_{\mathbb{R}}} = 0.$$

Determine $\xi_{1,1}$ and μ_1 in terms of $A(p^2)$ and $B(p^2)$ and their derivatives.