Statistical Physics
Exercise Sheet 4

Exercise 4.1 The Linear Boltzmann equation

The subject of this exercise is a Boltzmann equation no longer emerging from two-particle scattering, as discussed in class, but from scattering of single particles by static impurities. We assume that the particles do not interact among themselves, but are influenced by a static background of impurities which causes a particle of momentum $\vec{p}$ to be scattered to a state with new momentum $\vec{p}\,'$. We assume that the scattering happens within a negligibly short time interval and that it is elastic. Furthermore, we assume that each static scatterer is isotropic.

(a) Express these ideas mathematically: Find the relevant quantity describing the scattering, along with any of its symmetries, and write down the ensuing equation for $f(\vec{r}, \vec{p}, t)$.

(b) List all quantities φ conserved in the scattering, and prove the conservation law

$$\int d^3p \; \varphi(\vec{p}) \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} (\vec{r}, \vec{p}, t) \;=\; 0\,.$$

(c) State and prove the "H-theorem" for the linear Boltzmann equation. When is H constant in time? How can this H-theorem be understood physically?

Exercise 4.2 Local equilibrium state of a gas in a periodic potential

We consider a gas of N particles trapped in a box, $\vec{r} \in V = [0, L]^3$, in the presence of a conservative force $\vec{F}(\vec{r}) = -\nabla V(\vec{r})$ originating from a periodic potential in the x direction

$$V(\vec{r}) = V_0 \cos\left(\frac{2\pi x k}{L}\right), \quad k \in \mathbb{N}. \tag{1}$$

In the equilibrium the distribution function is given by

$$f_0(\vec{r}, \vec{p}) = \frac{n(\vec{r})}{(2\pi m k_B T)^{3/2}} e^{-\beta \frac{p^2}{2m}}, \quad \beta = \frac{1}{k_B T}. \tag{2}$$

a) Find the local density $n(\vec{r})$. Discuss the limits $V_0 \ll k_B T$ and $V_0 \gg k_B T$.

b) Determine the internal energy $U = \langle p^2/2m + V(\vec{r})\rangle$ and the specific heat $C_V = (\partial U/\partial T)_V$. Discuss these expressions in the limits $V_0 \ll k_B T$ and $V_0 \gg k_B T$.

c) Calculate the entropy $S(T, V, N)$.

Hints: The integral representation and the series expansion of the modified Bessel functions of the first kind for integer order n are

$$I_n(z) = \frac{1}{\pi} \int_0^{\pi} e^{z\cos\theta} \cos(n\theta) d\theta = \left(\frac{z}{2}\right)^n \sum_{k \geq 0} \frac{(z^2/4)^k}{k!(n+k)!}, \quad n \in \mathbb{Z}. \tag{3}$$

The asymptotic behavior for $z \to \infty$ is

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left(1 - \frac{4n^2 - 1}{8z} + \dots\right).$$

Furthermore, the relation $I_0'(z) = I_1(z)$ holds.