Statistical Physics
Exercise Sheet 7

Exercise 7.1 Ideal Ising Paramagnet II

Consider again the system of spins in a magnetic field h given in ex. 6.3, described by the Hamiltonian

$$\mathcal{H} = -\sum_{n=1}^{N} h s_i, \qquad s_i = \pm s \tag{1}$$

in the canonical ensemble.

a) Calculate the entropy $S(T, h)$ and discuss the behavior for the limits $T \to 0, \infty$ for finite h and for $h = 0$.
Show qualitatively that it is possible to cool this system by repeatedly isothermally increasing the field and then adiabatically decreasing it (adiabatic demagnetization). Is it theoretically possible to reach absolute zero temperature like this?

b) We now consider a system similar to the one described by eq. (1) where the spins, however, feel an effective magnetic field given by the applied field plus the magnetization of the system,

$$\mathcal{H} = -\sum_{n=1}^{N} s_i(h + \gamma m), \tag{2}$$

with m the magnetization,

$$m = \frac{1}{N}\sum_i \langle s_i \rangle \tag{3}$$

and $\gamma > 0$ some coupling constant. Calculate the magnetization for an applied magnetic field. Show that there are two temperature regimes $T \gtrless T_c$ where the magnetization vanishes / stays finite when turning off the field. What is T_c and how does the magnetization behave close to $T = 0$ and $T = T_c$? What happens with the susceptibility at $T = T_c$?
Hint: Calculate $h = h(m, T)$ and plot it for different T.

Exercise 7.2 Exact solution of the Ising chain

Consider the one-dimensional Ising model on a chain of length N with free ends

$$\mathcal{H} = -J\sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}, \quad \sigma_i = \pm 1. \tag{4}$$

a) Compute the partition function Z_N using a recursive procedure.

b) Calculate the magnetization density $m = \langle \sigma_j \rangle$ where the spin σ_j is far away from the ends.

c) Show that the *spin correlation function* $\Gamma_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$ decays exponentially on the scale of the *correlation length* $\xi = -[\log(\tanh \beta J)]^{-1}$ with increasing the distance $|j - i|$. What happens in the limit $T \to 0$?

d) Use the fluctuation relation in the thermodynamic limit $N \to \infty$,

$$\chi(T) = \frac{1}{k_B T} \left(\sum_j \langle \sigma_0 \sigma_j \rangle - N \langle \sigma_0 \rangle^2 \right) , \tag{5}$$

to calculate the magnetic susceptibility in zero magnetic field.