Statistical Physics
Exercise Sheet 10

Exercise 10.1 Greenhouse Effect

(a) Calculate the solar constant S_0 (energy flow density of the radiation of the sun on earth) using the following data: Temperature of the sun $T_S = 5800K$, radius of the sun $r_S = 6.96 \cdot 10^8 m$, distance sun-earth $R = 1.50 \cdot 10^{11} m$.

(b) Using the result of (a), calculate the earth's mean temperature. Model the earth as a black body and include the effect of reflection of the sun's radiation by modifying $S_0 \to (1-r)S_0$. Consider the cases $r = 0$ and $r = 0.3$.

(c) Building upon (b), include the greenhouse effect by modeling the atmosphere as a layer around earth that is completely transparent for the sun's radiation, but absorbs all the radiation from earth (like the glass roof of a greenhouse).

Exercise 10.2 Magnetostriction in a Spin-Dimer-Model

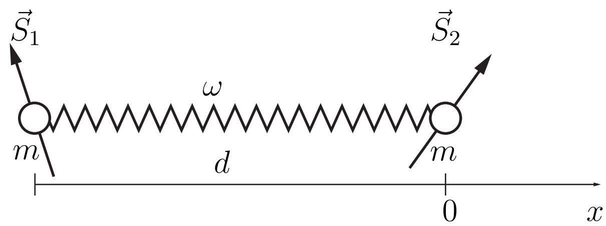
As in exercise 8.1, we again start with a dimer consisting of two (quantum) spins, $s = 1/2$, described by the Hamiltonian

$$\mathcal{H}_0 = J(\vec{S}_1 \cdot \vec{S}_2 + 3/4), \tag{1}$$

with $J > 0$. This time, however, the distance between the spins is not fixed but they are connected by a spring (cf. fig.) such that the Hamiltonian of the system reads

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 + J(1 - \lambda\hat{x})(\vec{S}_1 \cdot \vec{S}_2 + 3/4); \tag{2}$$

i.e., the spin-coupling constant depends on the distance between the two sites.



In the above figure, m is the mass of the two constituents, ω is the spring constant and d denotes the equilibrium distance between the two spins for the case of no spin-spin interaction. Furthermore, x is measured from this equilibrium.

(a) Write the Hamiltonian (2) in second quantized form and calculate the partition sum, the internal energy, the specific heat and the entropy. In the limit $T \to 0$, discuss the entropy for different values of λ.
Hint: Introduce an operator $\hat{n}_t$ defined through

$$\langle\sigma|\hat{n}_t|\sigma\rangle = \begin{cases} 1 & \sigma \text{ is a triplet,} \\ 0 & \sigma \text{ is the singlet,} \end{cases} \tag{3}$$

and trace first over the spin-degrees of freedom.

(b) Calculate the expectation value of the distance of the two spins, $\langle d + \hat{x} \rangle$, as well as the fluctuations $\langle (d + \hat{x})^2 \rangle$.
How can we manipulate this quantities by applying a magnetic field in z-direction, leading to an additional term in (2),

$$\mathcal{H}_m = -g\mu_B H \sum_{i,m} S^z_{i,m}? \tag{4}$$

(c) If the two sites carry an equal but opposite charge $\pm q$, the dimer forms a dipole with moment $P = q\langle d + x \rangle$. This dipole moment can be measured by applying an electric field E in x direction,

$$\mathcal{H}_{el} = -q(d + \hat{x}) \cdot E. \tag{5}$$

Calculate the zero-field susceptibility of the dimer,

$$\chi_0^{(el)} = -\left.\frac{\partial^2 F}{\partial E^2}\right|_{E=0}, \tag{6}$$

and compare with the result of the fluctuation-dissipation theorem,

$$\chi_0^{(el)} \propto \left(\left\langle (d + x)^2 \right\rangle - \left\langle d + x \right\rangle^2 \right). \tag{7}$$

Plot the zero-field susceptibility as a function of the applied magnetic field H and discuss.