Sheet II

Due: week of October 5

Question 1 [*Manifold* S^2]:

i) Show that the 2-sphere, i.e. the surface

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}\,,$$

is a differentiable manifold. To this end construct charts $\psi_i^\pm$ on $\{\pm x_i > 0\}$, and show that the transition functions are smooth (Hint: It is sufficient to show this for just one transition function).

ii) For this part use only the chart $\psi_1^+ : (x_1, x_2, x_3) \mapsto (u, v)$. Find the components a^μ and b^μ of the two basis vectors

$$X_u = \frac{\partial}{\partial u} = a^\mu \frac{\partial}{\partial x^\mu}\,, \qquad X_v = \frac{\partial}{\partial v} = b^\mu \frac{\partial}{\partial x^\mu}\,, \qquad \mu = 1, 2, 3$$

w.r.t. the partial derivatives of $\mathbb{R}^3$ by calculating $X_u(f|_{S^2})$ and $X_v(f|_{S^2})$ at a point $p \in S^2$, where f is a differentiable function on $\mathbb{R}^3$, i.e. calculate

$$X_u(f|_{S^2}) = \frac{\partial}{\partial u}(f \circ (\psi_1^+)^{-1})\,, \qquad X_v(f|_{S^2}) = \frac{\partial}{\partial v}(f \circ (\psi_1^+)^{-1})\,.$$

Furthermore, find the integral curves of the two basis vector fields by solving the equations

$$\dot{\gamma}_u(t) = X_u(\gamma_u(t))\,, \qquad \dot{\gamma}_v(t) = X_v(\gamma_v(t))$$

for the differentiable curves

$$\gamma_u(t) = (\gamma_{u1}(t), \gamma_{u2}(t), \gamma_{u3}(t))\,, \qquad \gamma_v(t) = (\gamma_{v1}(t), \gamma_{v2}(t), \gamma_{v3}(t))\,.$$

Question 2 [*Change of Basis in Tangent and Cotangent space*]:

In the chart defined by the coordinate functions x^μ, the coordinate basis for the tangent space T_p is defined by $X_\mu = \partial_\mu$, and the corresponding dual basis of the cotangent space T_p^* is given by $\mathrm{d}x^\mu$.

i) For a different chart, described by $\tilde{x}^\mu$, express the corresponding basis vectors $\widetilde{X}_\mu$ and $\mathrm{d}\tilde{x}^\mu$ in terms of X_μ and $\mathrm{d}x^\mu$, respectively. What is the transformation law of the corresponding components, i.e. writing $X = a^\mu X_\mu$ and $\omega = b_\nu \mathrm{d}x^\nu$, what is the transformation law for the coefficients a^μ and b_ν?

ii) Show that the operation of contraction C of a tensor

$$(CT)^{\mu_1,\ldots\mu_k}_{\nu_1,\ldots,\nu_l} = T^{\mu_1,\ldots,\mu_k,\sigma}_{\nu_1,\ldots,\nu_l,\sigma}$$

is independent of the choice of basis.

Question 3 [*Vector Fields*]:

A vector field is a linear map on the space of differentiable functions $\mathcal{F}$

$$X : \mathcal{F} \to \mathcal{F}$$

satisfying the derivation property,

$$X(fg) = X(f)\,g + f X(g)\ .$$

i) Show that $X \circ Y$ and $Y \circ X$ are *not* vector fields, but that the commutator

$$[X, Y] = X \circ Y - Y \circ X$$

is a vector field.

ii) Confirm that the commutator also satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\ .$$