Sheet III

Due: week of 12 October

Question 1 [*Symmetries of tensors*]:

The Riemann tensor $R_{\mu\nu\rho\lambda}$ has the properties

$$R_{\mu\nu\rho\lambda} = -R_{\nu\mu\rho\lambda} \ , \qquad R_{[\mu\nu\rho]\lambda} = 0 \ , \qquad R_{\mu\nu\rho\lambda} = -R_{\mu\nu\lambda\rho} \ .$$

(i) Show that it satisfies

$$R_{\mu\nu\rho\lambda} = R_{\rho\lambda\mu\nu} \ .$$

(ii) The Ricci tensor is defined by

$$R_{\mu\nu} = R_{\mu\rho\nu\lambda}\, g^{\rho\lambda} \ .$$

Show that the Ricci tensor is symmetric,

$$R_{\mu\nu} = R_{\nu\mu} \ .$$

(iii) For $n > 2$ we define the Weyl tensor $C_{\mu\nu\rho\lambda}$ by the equation

$$R_{\mu\nu\rho\lambda} = C_{\mu\nu\rho\lambda} + \frac{2}{n-2}\Big(g_{\mu[\rho}R_{\lambda]\nu} - g_{\nu[\rho}R_{\lambda]\mu}\Big) - \frac{2}{(n-1)(n-2)} R g_{\mu[\rho}g_{\lambda]\nu} \ ,$$

where R is the scalar curvature defined by

$$R = R_{\mu\nu} g^{\mu\nu} \ .$$

Show that the Weyl tensor has the same symmetry properties as the Riemann tensor, *i.e.*

$$C_{\mu\nu\rho\lambda} = -C_{\nu\mu\rho\lambda} \ , \qquad C_{[\mu\nu\rho]\lambda} = 0 \ , \qquad C_{\mu\nu\rho\lambda} = -C_{\mu\nu\lambda\rho} \ .$$

Furthermore, show that the Weyl tensor is traceless with respect to the contraction of any pair of indices.

Question 2 [*Lie derivative*]:

In components, the action of the Lie derivative L_X on a vector field R is given as

$$(L_X R)^\mu = \frac{\partial R^\mu}{\partial x^\nu} X^\nu - R^\nu \frac{\partial X^\mu}{\partial x^\nu} \ ,$$

while the action on a 1-form ω is

$$(L_X \omega)_\mu = \frac{\partial \omega_\mu}{\partial x^\nu} X^\nu + \omega_\nu \frac{\partial X^\nu}{\partial x^\mu} \ .$$

(i) Show that

$$L_X(Y) = [X, Y] \ .$$

(ii) Check that acting on vector fields and 1-forms we have

$$L_{[X,Y]} = L_X L_Y - L_Y L_X \ .$$

Question 3 [*Differential forms*]:

Check that the exterior derivative of a 2-form Ω, defined by

$$\begin{aligned} \mathrm{d}\Omega(X_1, X_2, X_3) =& X_1(\Omega(X_2, X_3)) - X_2(\Omega(X_1, X_3)) + X_3(\Omega(X_1, X_2)) \\ & -\Omega([X_1, X_2], X_3) + \Omega([X_1, X_3], X_2) - \Omega([X_2, X_3], X_1) \end{aligned}$$

defines indeed a 3-form, *i.e.* that

$$\mathrm{d}\Omega(fX_1, X_2, X_3) = \mathrm{d}\Omega(X_1, fX_2, X_3) = \mathrm{d}\Omega(X_1, X_2, fX_3) = f\mathrm{d}\Omega(X_1, X_2, X_3) \ .$$