Sheet V

Due: week of October 26

Question 1 [*Christoffel Symbols and Geodesic Equation in Euclidean Space*]:
The metric of Euclidean $\mathbb{R}^3$ in spherical coordinates is

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ .$$

i) Calculate the Christoffel components $\Gamma^\sigma_{\mu\nu}$ in this coordinate system.

ii) Write down the components of the geodesic equation in this coordinate system, and verify that the solutions correspond to straight lines in Cartesian coordinates.

Question 2 [*Riemann and Weyl Tensor*]:

i) Show that in two dimensions, the Riemann tensor takes the form

$$R_{abcd} = R g_{a[c} g_{d]b} \ .$$

(Hint: First deduce that the Riemann tensor has only one independent component in two dimensions. Then show that $g_{a[c}g_{d]b}$ spans the vector space of tensors having the symmetries of the Riemann tensor.)

ii) By similar arguments show that in three dimensions the Weyl tensor

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{2}{(n-1)(n-2)} R g_{a[c}g_{d]b}$$

vanishes identically. (Hint: In three dimensions the Riemann tensor has 6 independent components. Write down the different components of the Riemann tensor and express them in terms of the Ricci tensor R_{ab} and the scalar curcature R.)

Question 3 [*Metric and Riemann Tensor of a Sphere*]:

i) Determine the metric on the surface of a sphere of radius r in the usual spherical coordinates (θ, ϕ). Determine also the inverse metric $g^{\alpha\beta}$.

ii) Calculate the Riemann curvature tensor of the sphere. (Hint: Because of Question 2 i) there is only one independent component which you can take to be $R_{\theta\phi\theta\phi}$. Determine all other components in terms of it.)