Sheet VI

Due: week of 2 November

Question 1 [*Killing vector field*]:

(i) On sheet III, Question 2(i) we showed that

$$L_X Y = [X, Y] ,$$

and in the lecture we proved that

$$[X, Y] = X^a \nabla_a Y - Y^a \nabla_a X .$$

Combining these two equations it therefore follows that

$$(L_X Y)^b = X^a \nabla_a Y^b - Y^a \nabla_a X^b .$$

Since the Lie derivative has the Leibnitz property and commutes with contractions (why?), deduce from this result the action of the Lie derivative on 1-forms

$$(L_X \omega)_b = X^a \nabla_a \omega_b + \omega_a \nabla_b X^a \tag{1}$$

and on 2-forms

$$(L_X T)_{ab} = X^c \nabla_c T_{ab} + T_{cb} \nabla_a X^c + T_{ac} \nabla_b X^c . \tag{2}$$

Hint: By definition, $L_X f = X f = X^a \nabla_a f$.

(ii) By expressing the difference of two covariant derivatives in terms of the tensor $C^a_{bc} = C^a_{cb}$ show that the formulae (1) & (2) are in fact independent of the choice of covariant derivative.

(iii) Suppose $\phi_t : M \to M$ is a one-parameter group of isometries, $\phi^*_t g = g$, where g is the metric on M. Show that the generating vector field X satisfies the Killing vector equation

$$\nabla_a X_b + \nabla_b X_a = 0 ,$$

where ∇_a is the covariant derivative with respect to which the metric is covariantly constant.

Question 2 [*Affine parametrisations of curves*]:

(i) A geodesic $\gamma(t)$ is characterised by the property that the tangent vector is parallely propagated along itself, *i.e.* that the tangent vector $T = \frac{d\gamma(t)}{dt}$ satisfies

$$T^a \nabla_a T^b = \alpha T^b , \tag{3}$$

where α is some constant. Show that one can always find a parametrisation of the curve $t \equiv t(s)$ so that (3) becomes

$$S^a \nabla_a S^b = 0 ,$$

where S is the tangent vector with respect to s. (The resulting parametrisation is called the affine parametrisation.)
Hint: Work in coordinates!

(ii) Let t be an affine parameter of a geodesic γ. Show that any other affine parameter s of γ takes the form $s = at + b$, where a and b are constants.

(iii) Let $\gamma_s(t)$ be a smooth one-parameter family of geodesics, *i.e.* for each $s \in \mathbb{R}$, $\gamma_s(t)$ is a geodesic parametrised by an affine parameter t. The vector field $X = \frac{\partial}{\partial s}$ represents the displacement of nearby geodesics and is called the deviation vector. Because of (ii) there is a 'gauge freedom' in the definition of X since we can change the t-parametrisations in an s-dependent manner, *i.e.*

$$t \mapsto t' = a(s)t + b(s) \ .$$

Show that this modifies X by adding to it a multiple of $T = \frac{\partial}{\partial t}$. For the case where the geodesics are timelike or spacelike show that we can use this gauge freedom to choose X^a always orthogonal to T^b, *i.e.*

$$g_{ab} X^a T^b = 0 \ .$$

Question 3 [*Inverse metric*]:

Use the formula for the inverse of a matrix to show that

$$g^{\nu\sigma} \partial_\mu g_{\nu\sigma} = \frac{1}{g} \frac{\partial g}{\partial x^\mu} \ ,$$

where $g = \det(g_{\mu\nu})$.
Hint: The determinant depends on x^μ via the matrix elements $g_{\mu\nu}$. Use column expansion!