Sheet VII

Due: week of November 9

Question 1 [*Curvature Tensor*]:

Consider a two-dimensional manifold whose metric is of the form

$$ds^2 = \Omega^2(x,t)(-dt^2 + dx^2) \ . \tag{1}$$

Calculate the Riemann curvature tensor of this metric

i) directly by coordinate basis methods; and

ii) by using the tetrad methods introduced in the lecture.

Question 2 [*Electrodynamics*]:

The general covariant form of Maxwell equations is

$$\nabla^a F_{ab} = -4\pi j_b \ , \qquad \nabla_{[a} F_{bc]} = 0 \, . \tag{2}$$

Show that these equations imply charge conservation

$$\nabla_a j^a = 0 \tag{3}$$

i) by making use of the antisymmetry $F_{ab} = -F_{ba}$;

ii) by writing F_{ab} in terms of a potential A_b,

$$F_{ab} = \nabla_a A_b - \nabla_b A_a \, , \tag{4}$$

and imposing the covariant Lorentz gauge condition $\nabla^a A_a = 0$.

Question 3 [*Bel-Robinson Tensor*]:

In general relativity no meaningful expression is known for the local stress-energy tensor of the gravitational field. However, a four-index tensor T_{abcd} can be constructed out of the curvature in a manner closely analogous to the way in which the stress tensor of the electromagnetic field is constructed out of F_{ab}. The resulting tensor is the Bel-Robinson tensor which, in terms of the Weyl tensor C_{abcd}, is defined by

$$T_{abcd} = C_{aecf} C_b{}^e{}_d{}^f + \frac{1}{4} \epsilon_{ae}{}^{hi} \epsilon_b{}^{ej}{}_k C_{hicf} C_j{}^k{}_d{}^f \ , \tag{5}$$

where ϵ_{abcd} is the totally antisymmetric tensor in four dimensions with $\epsilon^{0123} = +1$.

i) Show that

$$\epsilon^{abcd} \epsilon_{aefg} = -6 \, \delta^{[b}{}_e \delta^c{}_f \delta^{d]}{}_g \tag{6}$$

and use this relation to bring (5) into the alternative form

$$T_{abcd} = C_{aecf} C_b{}^e{}_d{}^f - \frac{3}{2} g_{a[b} C_{jk]cf} C^{jk}{}_d{}^f \ .$$

ii) It can be shown that $T_{abcd} = T_{(abcd)}$. Using this, show that

$$T^a{}_{acd} = 0 \; . \tag{7}$$

iii) Using the Bianchi identity

$$\nabla_{[a} R_{bc]d}{}^e = 0 \; , \tag{8}$$

show that for $R_{ab} = 0$ (vacuum) we have

$$\nabla^a T_{abcd} = 0 \; . \tag{9}$$