Homework 7 - Mechanics

To be handed in: Mon 09-11-09

1. **A Foretaste of General Relativity:** The Newtonian theory is just a limiting case of General Relativity. In spherical coordinates, the potential of the Kepler problem is modified by general relativity effects as follows

$$U(r) = -\frac{M}{r} + \frac{\alpha}{r^3} + O(\alpha^2). \tag{1}$$

Here α plays the role of a small perturbation parameter. Our units are chosen such, that $G = m = 1$.

a) Use the Lagrange formalism to derive the differential equation

$$u''(\phi) + u(\phi) = l^{-2}\left[M - 3\alpha u^2(\phi)\right] + O(\alpha^2) \tag{2}$$

for the variable $u = r^{-1}$

b) Solve equation (2) perturbatively up to first order in α. You should obtain

$$u = A_0 \sin\phi + A_1 \cos\phi + A_2 + \frac{1}{2}A_3\left(\phi\sin\phi + \frac{\epsilon}{2} - \frac{\epsilon}{6}\cos 2\phi\right) + O(\alpha^2) \tag{3}$$

with $A_2 = Ml^{-2} - 3\alpha M^2 l^{-6}$ and $A_3 = -6\alpha\epsilon M^2 l^{-6}$. The constants A_0 and A_1 are determined by the boundary conditions.

Hint: Find the solution at zero order ($\alpha = 0$) and insert it into the right hand side of equation (2).

c) Show that the shift of the perihelion is given by

$$\Delta\phi = -\frac{6\pi\alpha}{Ma^2(1-\epsilon^2)^2} + O(\alpha^2). \tag{4}$$

2. **Amplitude of Oscillation:** Consider a one-dimensional oscillation in a convex potential $V''(x) > 0$ with minimum $V(0) = 0$. For a particle of mass m and energy $0 < E < E_0$, let $T(E)$ be the period of the oscillation. Calculate the distance between the points of reversal

$$d(E) = x_2(E) - x_1(E). \tag{5}$$

Hint: Find an expression for $T(E)$ and calculate the integral

$$\int_0^F \frac{T(E)dE}{\sqrt{F-E}}. \tag{6}$$