# Solutions 2 - Newtonian mechanics and Euler-Lagrange formalism

September 25, 2009

**1. Reminder of Newtonian mechanics**

Answer:

(a) The equation of motion for the center of mass simply reads

$$z_{\mathrm{cm}}(t) = \frac{g}{2}t^2. \tag{1}$$

(b) At $t = 0$ (immediately after cutting the rope), the spring is still in equilibrium elongation.

$$\begin{aligned} m_1\ddot{z}_{1,l}(0) &= m_1 g + m_2 g & (2)\\ m_2\ddot{z}_{2,l}(0) &= 0, & (3)\end{aligned}$$

where the index $l$ denotes the lab system.

(c) We now switch to the accelerated center of mass frame as our frame of reference. There, the gravitational forces $m_i\vec{g}$ are canceled by the fictitious forces. Therefore, it can be regarded as a closed inertial system.

The spring exerts equal forces on both masses, such that in the center of mass the force balance reads

$$\begin{aligned} m_1\ddot{z}_1(t) &= -m_2\ddot{z}_2(t) & (4)\\ \Rightarrow \ddot{z}_2(t) &= -\frac{m_1}{m_2}\ddot{z}_1(t). & (5)\end{aligned}$$

Since the two masses are oscillating in push–pull mode, it follows that

$$z_2(t) = -\frac{m_1}{m_2}z_1(t). \tag{6}$$

Therefore,

$$m_1\ddot{z}_1 = D(z_2 - z_1) = -D\left(1 + \frac{m_1}{m_2}\right)z_1(t) \tag{7}$$

$$\Rightarrow \ddot{z}_1 + \frac{D}{m}z_1 = 0, \tag{8}$$

where we have defined the so–called reduced mass

$$m := \frac{m_1 m_2}{m_1 + m_2}. \tag{9}$$

The frequency $\omega = \sqrt{D/m}$ follows immediately from the ansatz

$$z_1(t) = A_1 \cos(\omega t - \phi_1). \tag{10}$$

(d) From equation (6) we know that

$$A_2 = \frac{m_1}{m_2} A_1. \tag{11}$$

Furthermore, the initial elongation $\Delta s$ equals the sum of the oscillation amplitudes:

$$\Delta s = \frac{m_2 g}{D} = A_1 + A_2. \tag{12}$$

Equations (11) and (12) then lead to

$$A_1 = \frac{mg}{D}\frac{m_2}{m_1}, \tag{13}$$

$$A_2 = \frac{mg}{D}. \tag{14}$$

$$\tag{15}$$

**2. Chopper carrying load on a rope – a pendulum with moving pivot:**

Answer:

We choose the generalized variables $x$ and $\phi$. The pivot has the coordinates $\mathbf{r}_1 = (x, 0, 0)$ while the oscillating point mass is given by $\mathbf{r}_2 = (x + l\sin\phi, 0, -l\cos\phi)$. The kinetic and potential energies are

$$T = \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2$$
$$V = m_2 g \mathbf{r}_2 \cdot \mathbf{e}_z$$

(a) We substitute $\mathbf{r}_i$ by their parametrization and find for the Lagrangian

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2 l \dot{x}\dot{\phi}\cos\phi + \frac{1}{2} m_2 l^2 \dot{\phi}^2 + m_2 g l \cos\phi \,. \end{aligned}$$

(b) We determine now the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

for $q_1 = x$ and $q_2 = \phi$.

(a) $q_1 = x$ :

$$\frac{\partial L}{\partial x} = 0 \ , \tag{16}$$

$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\,\dot{x} + m_2 l \dot{\phi} \cos\phi \ . \tag{17}$$

The quantity of equation (17) (the canonical momentum) is conserved due to equation (16). Thus we can write

$$\dot{x} = \frac{P - m_2 l \dot{\phi} \cos\phi}{m_1 + m_2} \ , \tag{18}$$

where $P$ is the constant value of (16).

(b) $q_2 = \phi$ :

$$\frac{\partial L}{\partial \phi} = -m_2 l \sin\phi \left( g + \dot{x}\dot{\phi} \right) \ , \tag{19}$$

$$\frac{\partial L}{\partial \dot{x}} = m_2 l \dot{x} \cos\phi + m_2 l^2 \dot{\phi} \ . \tag{20}$$

After dropping a factor $m_2 l$ we find the equation of motion

$$l\ddot{\phi} = -\ddot{x}\cos\phi - g\sin\phi \ . \tag{21}$$

(c) We can eliminate $\ddot{x}$ from (21) using (18) and get:

$$l\ddot{\phi}\left(1 - \frac{m_1}{m_1 + m_2}\cos^2\phi\right) + \frac{m_2}{m_1 + m_2}\dot{\phi}^2 \sin\phi\cos\phi = -g\sin\phi \ .$$

The initial conditions at time $t_0$ are

$$x(t_0) = x_0 \ , \qquad \phi(t_0) = \phi_0 \ ,$$
$$\dot{x}(t_0) = v_0 \ , \qquad \dot{\phi}(t_0) = \omega_0 \ .$$

For small displacements we have $\cos\phi \approx 1$ and $\sin\phi \approx \phi$. We find

$$\ddot{x} = -l\ddot{\phi} - g\phi \ , \tag{22}$$

$$\ddot{\phi} = -\frac{(m_1 + m_2)g}{m_1 l}\phi - \frac{m_2}{m_1}\dot{\phi}^2\phi \ . \tag{23}$$

If we linearize the second equation (assuming $\left|\frac{m_2}{m_1}\dot{\phi}^2\right| << \frac{(m_1+m_2)g}{m_1 l}$), we recognize the equation of a harmonic oscillation with solution

$$\phi(t) = A\cos\Omega(t - t_0) + B\sin\Omega(t - t_0) \qquad \text{with} \qquad \Omega = \sqrt{\frac{(m_1 + m_2)g}{m_1 l}} \ .$$

The coefficients $A$ and $B$ are to be determined from the initial conditions:

$$\phi(t_0) = A \overset{!}{=} \phi_0 \ ,$$
$$\dot{\phi}(t_0) = B\Omega \overset{!}{=} \omega_0 \ .$$

Thus we we find $A = \phi_0$ and $B = \frac{\omega_0}{\Omega}$. We plug this solution into (22) and find

$$\dot{x} = \frac{P}{m_1 + m_2} - \frac{m_2 l}{m_1 + m_2} \left(-\Omega\phi_0 \sin\Omega(t - t_0) + \omega_0 \cos\Omega(t - t_0)\right)$$
$$\Rightarrow x = x_0 + v_0 t - \frac{g m_2}{m_1 \Omega^2} \left(\phi_0 \cos\Omega(t - t_0) + \frac{\omega_0}{\Omega} \sin\Omega(t - t_0)\right) \ ,$$

where $v_0 = \frac{P}{m_1+m_2}$.

(d) The forces of constraints follow from

$$\mathbf{F}'_i = m_i \ddot{\mathbf{r}}_i - \mathbf{F}_{G,i} \ ,$$

where $\mathbf{F}_{G,i}$ is the gravitational force acting on particle $i$ and $\mathbf{r}_i$ is the trajectory of particle $i$ as follows from the calculation above.