# Mechanics Fall 2009, Solutions 8

1. **Blasting of an industrial chimney**
   Since the chimney is rotating around 0 we get the equation of motion

$$I \cdot \ddot{\varphi} = Mg\frac{L}{2}\sin\varphi$$

where

$$I = \frac{ML^2}{3}$$

is the moment of inertia of the the chimney. We therefore have an angular acceleration of

$$\ddot{\varphi} = \frac{3}{2}\frac{g\sin\varphi}{L}$$

In the frame of the falling chimney an element $dm = \frac{M}{L}dx$ experiences a perpendicular force given by

$$dF = -\frac{M}{L}x\ddot{\varphi}dx + \frac{M}{L}g\sin\varphi dx$$

An element at distance $x_0$ from 0 experiences a torque due to the rest of the chimney above it. This torque is given by

$$\begin{aligned} D(x_0) &= \frac{M}{L}\ddot{\varphi}\int_{x_0}^{L} x(x-x_0)dx - \frac{M}{L}g\sin\varphi\int_{x_0}^{L}(x-x_0)dx \\ &= \frac{M}{L}\left[\ddot{\varphi}\left(\frac{x_0^3}{6} - \frac{L^2}{2}x_0\right) + g\sin\varphi\left(Lx_0 - \frac{x_0^2}{2}\right)\right] + const \end{aligned}$$

The extremal torque is found by

$$\frac{dD(x_0)}{dx_0} = 0$$

with the solutions $x_0 = L/3$ (maximal torque) and $x_0 = L$ (minimal torque). The maximum torque is therefore found at distance $x_0 = L/3$ from 0. This is where the chimney might possibly break.

2. **Moments of Inertia**

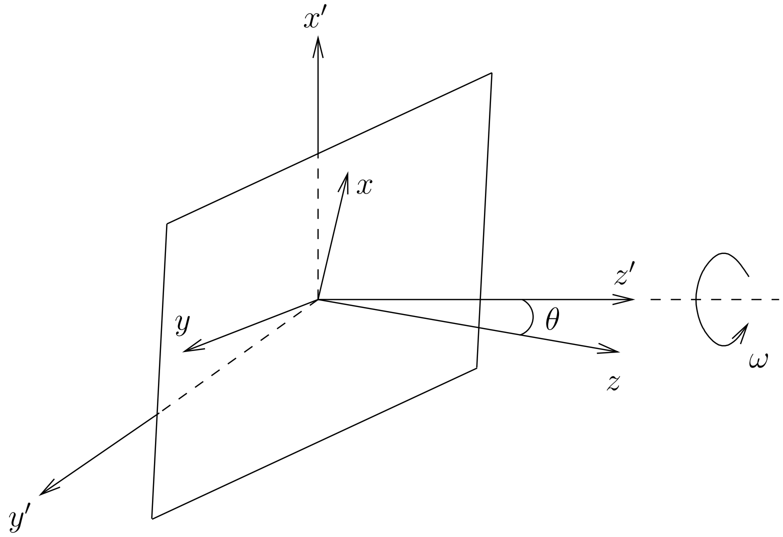


Figure 1: Thin square plate

(a) Take the origin at the centre $O$ of the square. For a coordinate frame attached to the square, take the plane of the square as the $xy$-plane with the $x$- and $y$-axis parallel to the sides. The $z$-axis, which is along the normal, makes an angle $\theta$ with the $z'$-axis of the laboratory frame about which the square rotates, as shown in figure 1. We also assume that the $x$-, $z$- and $z'$-axes are coplanar.

Then by symmetry the $x$-, $y$- and $z$-axes are the principal axes of inertia about $O$, with corresponding moments of inertia

$$I_{xx} = I_{yy} = \frac{ma^2}{12}, \qquad I_{zz} = \frac{ma^2}{6},$$

where $m$ is the mass of the square.

(b) The angular momentum $\mathbf{J}$ resolved along the rotating frame coordinate axes is

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \frac{ma^2}{12} & 0 & 0 \\ 0 & \frac{ma^2}{12} & 0 \\ 0 & 0 & \frac{ma^2}{6} \end{pmatrix} \begin{pmatrix} \omega \sin\theta \\ 0 \\ \omega \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{ma^2}{12}\omega \sin\theta \\ 0 \\ \frac{ma^2}{6}\omega \sin\theta \end{pmatrix}.$$

We can choose the laboratory frame so that its $y'$-axis coincides witht the $y$-axis at $t = 0$. Then the unit vectors of the two frames are related by

$$
\begin{aligned}
\mathbf{e}_x &= \cos\theta\cos\omega t\mathbf{e}_{x'} + \cos\theta\sin\omega t\mathbf{e}_{y'} + \sin\theta\mathbf{e}_{z'}, \\
\mathbf{e}_y &= -\sin\omega t\mathbf{e}_{x'} + \cos\omega t\mathbf{e}_{y'}, \\
\mathbf{e}_z &= -\sin\theta\cos\omega t\mathbf{e}_{x'} - \sin\theta\sin\omega t\mathbf{e}_{y'} + \cos\theta\mathbf{e}^{z'}.
\end{aligned}
$$

Hence the angular momentum resolved along the laboratory frame coordinate axes is

$$
\begin{aligned}
\begin{pmatrix} J_{x'} \\ J_{y'} \\ J_{z'} \end{pmatrix} &= \begin{pmatrix} \cos\theta\cos\omega t & -\sin\omega t & -\sin\theta\cos\omega t \\ \cos\theta\sin\omega t & \cos\omega t & -\sin\theta\sin\omega t \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{ma^2}{12}\omega\sin\theta \\ 0 \\ \frac{ma^2}{6}\omega\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} -\frac{ma^2}{12}\omega\sin\theta\cos\theta\cos\omega t \\ -\frac{ma^2}{12}\omega\sin\theta\cos\theta\sin\omega t \\ \frac{ma^2}{12}\omega(1+\cos^2\theta) \end{pmatrix}.
\end{aligned}
$$

(c) The torque on the axis in the laboratory frame is given by

$$
\mathbf{M} = \frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = \begin{pmatrix} \frac{ma^2}{12}\omega^2\sin\theta\cos\theta\sin\omega t \\ -\frac{ma^2}{12}\omega^2\sin\theta\cos\theta\cos\omega t \\ 0 \end{pmatrix}.
$$