Solutions 10: Hamiltonian formalism

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1. Reviewed harmonic oscillator

a) (i) The Poisson bracket of the new variables with respect to the old is

$$\{Q,P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \cos\theta\cos\theta - \left(-\frac{1}{m\omega}\sin\theta\right)(m\omega\sin\theta) = 1. \quad (1)$$

Since it equals 1, the transformation from (q,p) to (Q,P) is canonical.

(ii) Suppose that we regard this as a type i transformation, taking the coordinates q and Q as the independent variables. The momenta p and P are then given by

$$p = m\omega\left(q\cot\theta - \frac{Q}{\sin\theta}\right), \qquad P = m\omega\left(\frac{q}{\sin\theta} - Q\cot\theta\right). \quad (2)$$

Now consider the differential form

$$pdq - PdQ = m\omega\left(q\cot\theta - \frac{Q}{\sin\theta}\right)dq - m\omega\left(\frac{q}{\sin\theta} - Q\cot\theta\right)dQ \quad (3)$$

$$= d\left(\frac{1}{2}m\omega(q^2+Q^2)\cot\theta - m\omega\frac{qQ}{\sin\theta}\right). \quad (4)$$

Since it is an exact differential, this again shows that the transformation is canonical. The type 1 generating function is

$$F_1(q,Q) = \frac{1}{2}m\omega(q^2+Q^2)\cot\theta - m\omega\frac{qQ}{\sin\theta}. \quad (5)$$

b) The type 2 generating function $F_2(q,P)$ can be obtained by setting

$$\begin{aligned} F_2 &= F_1 + PQ && (6)\\ &= \frac{1}{2}m\omega(q^2 - Q^2)\cot\theta. && (7) \end{aligned}$$

We must still express this in terms of the appropriate type 2 variables, q and P, by setting

$$Q = \frac{q}{\cos\theta} - \frac{P}{m\omega}\tan\theta. \tag{8}$$

This gives

$$F_2(q,P) = \frac{qP}{\cos\theta} - \frac{1}{2}m\omega(q^2 + \frac{P^2}{m^2\omega^2})\tan\theta. \tag{9}$$

To check this expression, we evaluate its derivatives

$$\frac{\partial F_2*}{\partial q}_P = \frac{P}{\cos\theta} - m\omega\tan\theta = p\ , \qquad \frac{\partial F_2}{\partial P}_q = \frac{q}{\cos\theta} - \frac{P}{m\omega}\tan\theta = Q. \tag{10}$$

c) (i) We introduce the new canonical variables (Q,P) by setting,

$$q = Q\cos\theta + \frac{P}{m\omega}\sin\theta\ , \qquad p = -m\omega Q\sin\theta + P\cos\theta. \tag{11}$$

The Hamiltonian for the new canonical variables is given by

$$K(Q,P,t) = H(q,p) + \left(\frac{\partial F_2}{\partial t}\right)_{q,P} \tag{12}$$

where $F_2(q,P,t)$ is the type 2 generating function of the transformation (evaluated previously). The first term in $K(Q,P,t)$ is the old Hamiltonian $H(q,p)$, expressed in terms of the new variables,

$$\begin{aligned} H(q,p) &= \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 && (13)\\ &= \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2 = H(Q,P). && (14) \end{aligned}$$

The second term in $K(Q,P,t)$ is the time derivative of the generating function,

$$\left(\frac{\partial F_2}{\partial t}\right)_{q,P} = \left[qP\sin\theta - \frac{1}{2}m\omega\left(q^2 + \frac{P^2}{m^2\omega^2}\right)\right]\frac{\dot\theta}{\cos^2\theta} \tag{15}$$

$$= \left[qP\sin\theta - \frac{1}{2}m\omega\left(q^2 + \frac{P^2}{m^2\omega^2}\right)\right]\frac{\dot\theta}{\cos^2\theta} \tag{16}$$

$$= -\left(\frac{P^2}{2m\omega} + \frac{1}{2}m\omega Q^2\right)\dot\theta = -H(Q,P)(\dot\theta/\omega). \tag{17}$$

The new Hamiltonian is then

$$K(Q,P,t) = H(Q,P)(1-\dot\theta/\omega) \tag{18}$$

and reduces to zero if we take $\theta = \omega t$.

(ii) Hamilton's equation for the new variables are then

$$\frac{dQ}{dt} = \frac{\partial K}{\partial P} = 0\ , \qquad \frac{dP}{dt} = -\frac{\partial K}{\partial Q} = 0, \tag{19}$$

so the new canonical variables are constants $Q = Q_0$, $P = P_0$. The equations of the canonical transformation then give the original variables (q,p) as functions of time,

$$q(t) = Q_0\cos\omega t + \frac{P_0}{m\omega}\sin\omega t\ , \qquad p = -m\omega Q_0\sin\omega t + P_0\cos\omega t. \tag{20}$$

This is the well known solution to the harmonic oscillator problem. The new canonical variables (Q_0, P_0) are the initial $(t=0)$ values of the original variables (q,p).

2. Charged particle in a uniform magnetic field

a) Starting from the Hamiltonian and the definition of the vector potential we have

$$H = \frac{1}{2m}\left(|\vec p|^2 - \frac{2q}{c}\vec p\cdot\vec A + \frac{q^2}{c^2}|\vec A|^2\right) \tag{21}$$

$$= \frac{1}{2m}\left[(p_x^2+p_y^2+p_z^2) - \frac{B_0 q}{c}(xp_y - yp_x) + \frac{B_0^2 q^2}{4c^2}(x^2+y^2)\right] \tag{22}$$

Since

$$\frac{\partial H}{\partial z} = 0 \tag{23}$$

we know that $p_z \equiv constant$, so we can chose a frame in which $p_z \equiv 0$. The remaining Hamilton equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{2m}\left(2p_x + \frac{B_0 q}{c}y\right) \tag{24}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{1}{2m}\left(2p_y - \frac{B_0 q}{c}x\right) \tag{25}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{B_0 q}{2mc}\left(\frac{B_0 q}{2c}x - p_y\right) \tag{26}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{B_0 q}{2mc}\left(\frac{B_0 q}{2c}y + p_x\right). \tag{27}$$

Combining eqs. (24) and (27) we find that

$$\frac{d}{dt}\left(2p_y + \frac{B_0 q}{c}x\right) = 0 \quad \Longrightarrow \quad 2p_y = -\frac{B_0 q}{c}(x - 2x_0), \tag{28}$$

and from eqs. (25) and (26) we get

$$\frac{d}{dt}\left(2p_x - \frac{B_0 q}{c}y\right) = 0 \quad \Longrightarrow \quad 2p_x = \frac{B_0 q}{c}(y - 2y_0) \tag{29}$$

(we include a factor 2 in the integration constants to simplify a bit the following steps). Replacing we have

$$\dot{x} = \omega_c(y - y_0) \tag{30}$$

$$\dot{y} = -\omega_c(x - x_0), \tag{31}$$

and the solution to these first order linear equations is

$$x(t) = A\sin(\omega_c t + \phi) + x_0 \tag{32}$$

$$y(t) = A\cos(\omega_c t + \phi) + y_0. \tag{33}$$

Going back to eqs. (28) and (29) we finally find

$$p_x(t) = \frac{m\omega_c}{2}\left(A\cos(\omega_c t + \phi) - y_0\right) \tag{34}$$

$$p_y(t) = -\frac{m\omega_c}{2}\left(A\sin(\omega_c t + \phi) - x_0\right). \tag{35}$$

b) Going back to the original expresion of H and introducing the new coordinates and momenta we have

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 \tag{36}$$

$$= \frac{1}{2m}\left[\left(p_x + \frac{m\omega_c}{2}y\right)^2 + \left(p_y - \frac{m\omega_c}{2}x\right)^2\right] \tag{37}$$

$$= \frac{1}{2m}\left[\left(\sqrt{2m\omega_c p_1}\cos q_1\right)^2 + \left(\sqrt{2m\omega_c p_1}\sin q_1\right)^2\right] \tag{38}$$

$$= \omega_c p_1 \tag{39}$$

c) With the Hamiltonian expressed in terms of the new canonical variables there is just one non-trivial Hamilton equation:

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = \omega_c, \tag{40}$$

so

$$q_1(t) = \omega_c t + \phi. \tag{41}$$

The other variables, namely q_2, p_1, and p_2, are just constants. Going back to the old coordinates, we easily recover the expression given in eqs. (32), (33), (34) and (35).