Quantenmechanik I
Serie 9

Exercise 9.1 Minimum uncertainty wavefunction

We have seen that $\Delta x \Delta p \geq \hbar/2$. In this exercise we will see that the wave function that minimises this uncertainty in one dimension is a Gaussian.
Assume $\langle x \rangle = \langle p \rangle = 0$. Use the Schwarz inequality, $||f||^2||g||^2 \geq |(f,g)|^2$, with $f = x\psi$ and $g = p\psi$, to show that the equality only holds if $p\psi = \lambda x \psi$, where $\lambda \in \mathbb{C}$ is a constant, and $(\psi, (xp+px)\psi) = 0$.
Derive an expression for ψ, obtaining $\psi = A \exp\left(-\frac{|\lambda| x^2}{2\hbar}\right)$.

Exercise 9.2 Symmetry and projective representations – time translations

In quantum mechanics systems usually present some symmetries. Symmetries represent operations that leave certain properties of the state of the system unchanged. Such transformations form symmetry groups G. Each element $g \in G$ represents a unitary or antiunitary operator $U(g)$ that acts on elements of the Hilbert space,

$$g \longmapsto U(g) : \mathcal{H} \to \mathcal{H}.$$

The composition rule for symmetric operations is not as straightforward as one would wish, for in general there is a phase factor w to be considered,

$$U(g_1)U(g_2) = w(g_1, g_1)U(g_1 \cdot g_2), \quad |w(g_1, g_1)| = 1.$$

Representations where the phase is not trivial are called projective representations. In the following exercises we will see how this phase factor behaves for two important kinds of symmetries: time and space translations.
We start with the fist case, time translations. When a system is autonomous, ie, is ruled by a time independent Hamiltonian, time translations form a symmetry group.
This group is formed by the set $\mathbb{R}$ (since a system may be translated by any amount $t \in \mathbb{R}$ of time) and the usual sum (in principle one expects that moving the system by t_1 and then for t_2 is the same as moving it by $t_1 + t_2$ at once).
The projective representation for this group is given by

$$t \longmapsto U(t), \qquad U(t_1)U(t_2) = w(t_1, t_2)U(t_1 + t_2), \quad |w(t_1, t_1)| = 1.$$

a) Use the associativity of the product of operators $U(t)$ to prove the cocycle condition of the phase factor,

$$w(x, y)w(x + y, z) = w(x, y + z)w(y, z), \quad \forall x, y, z \in \mathbb{R}.$$

Check that in particular

$$w(x, 0) = w(0, z) = w(0, 0), \quad \forall x, z \in \mathbb{R}.$$

b) We want the phase to be always 1. For this we will try to find a gauge transformation on the operators U,

$$\tilde{U}(t) = \phi(t)U(t), \quad |\phi| = 1, \qquad t \longmapsto U(t) \longmapsto \tilde{U}(t),$$

such that

$$\tilde{U}(t_1)\tilde{U}(t_2) = \tilde{w}(t_1, t_2)\tilde{U}(t_1 + t_2) = \tilde{U}(t_1 + t_2).$$

Check that the necessary condition for that is

$$\tilde{w}(t_1, t_2) = w(t_1, t_2)\frac{\phi(t_1)\phi(t_2)}{\phi(t_1 + t_2)} = 1.$$

c) We now have to choose $\phi(t)$ carefully so that the condition above is verified. We start by imposing

$$w(x, 0) = w(0, z) = w(0, 0) = 1, \quad \forall x, z \in \mathbb{R},$$

by applying a preliminar gauge transformation $\Phi(t) = w(0, 0)^{-1}$. Check that this condition implies $\phi(0) = 1$ and verify that if $\phi(t)$ is a solution then $\phi(t)e^{i\alpha t}$, $\alpha \in \mathbb{R}$ is also a solution.

d) Since the solution is not unique, we may impose another condition, like $\phi'(0) = 0$. Assuming differentiability for w, verify that this extra condition implies

$$\frac{\partial}{\partial y} w(x,y)\bigg|_{y=0} = \frac{\partial}{\partial x} \ln \phi(x).$$

e) We may then define $\phi(t)$ as the solution of the system

$$\begin{cases} \frac{\partial}{\partial y} w(x,y)\Big|_{y=0} = \frac{\partial}{\partial x} \ln \phi(x), \\ \phi(0) = 1. \end{cases}$$

Check that this system always has a solution. This implies that independently of the form of the phase factor of a projective representation of a symmetry group isomorphic to $(\mathbb{R}, +)$, such as time translations, we may always make the phase vanish via a gauge transformation.

Exercise 9.3 Space translations in the plane

We will now verify that for symmetries generated by space translations in the plane we cannot get rid of the phase factor like we did for translations in one dimension.
Let the translation operators along the x and y axis be P_1 and P_2. To move a system through a vector $a\hat{e_x}$ along the x axis one applies the transformation $T(a,0) = e^{iaP_1}$. Similarly, for a translation of $b\hat{e_y}$, the operator to be applied is given by $T(0,b) = e^{ibP_2}$. Now, to move the system by $a\hat{e_x} + b\hat{e_y}$ one must apply

$$T(a,b) = e^{iaP_1} e^{ibP_2} = e^{i(aP_1 + bP_2)} w(a,b).$$

The phase $w(a,b)$ depends on the commutator of the two operators, $[P_1, P_2]$. We will see that this phase is not trivial for a very natural case: an electron moving in a magnetic field. We have seen before that the commutator between the generalised momenta was given by $[\Pi_x, \Pi_y] = i\alpha\mathbb{1}$. We will export that result to here and assume $[P_1, P_2] = i\alpha\mathbb{1}$.

a) Use the Baker-Campbell-Hausdorff formula,

$$e^A e^B = \exp(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]] + \dots),$$

to achieve the result $e^{iaP_1} e^{ibP_2} = e^{i(aP_1 + bP_2)} e^{-\frac{i\alpha ab}{2}}$.

b) Now say $\vec{r} = (a,b), \vec{r'} = (a', b')$. We have $T(\vec{r}) = e^{i(aP_1 + bP_2)} = e^{i\vec{r}\cdot\vec{P}}$, and we would like to define a gauge transformation

$$\tilde{T}(\vec{r}) = \phi(\vec{r}) e^{i\vec{r}\cdot\vec{P}} = e^{i\theta(\vec{r})} e^{i\vec{r}\cdot\vec{P}}$$

such that

$$\tilde{T}(\vec{r} + \vec{r'}) = \tilde{T}(\vec{r})\tilde{T}(\vec{r'}).$$

Prove that it is impossible to reach a gauge transformation with the desired properties.

Exercise 9.4 Unitary and antiunitary symmetries

a unitary operator U will act on the scalar product as $(U\phi, U\psi) = (\phi, \psi)$. On the other hand, an antiunitary operator A will act as $(A\phi, A\psi) = (\phi, \psi)^* = (\psi, \phi)$.

a) The time reversal operator acts as $Te^{-\frac{itH}{\hbar}}\psi = e^{\frac{itH}{\hbar}} T\psi$. Also, we know that $T^2 = 1$. Prove that T is antiunitary. Hint: assume that T is unitary and reach the condition that the spectrum of H equals the spectrum of $-H$.

b) The space reflection operator acts as $P\vec{x}\psi = -\vec{x}P\psi$. Again, $P^2 = 1$. Prove that P is unitary.