Quantenmechanik I
Serie 4

HS 09
Prof. J. Fröhlich

Exercise 4.1 Charged particle in an external electromagnetic field

The Lagrangian for a set of particles of masses m_i and of charges q_i in an electromagnetic field is

$$\mathcal{L}(x_i, \dot{x}_i) = \sum_i \frac{1}{2} m_i \dot{x}_i^2 - \sum_i q_i \varphi_i(x_i) - \sum_i \frac{q_i}{c} \dot{x}_i A(x_i),$$

where A and φ are the electromagnetic vector and scalar potentials respectively.

a) Compute the momenta p_i defined as

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i}$$

b) Compute the Hamiltonian

$$H(x_i, p_i) = \sum_i \dot{x}_i p_i - \mathcal{L}(x_i, \dot{x}_i)$$

and write it in terms of the x_i and p_i exclusively (rewrite $\dot{x}_i$ in terms of p_i if necessary).

c) Show that replacing p_i through $-i\hbar\frac{\partial}{\partial x_i}$, one recovers the Schrödinger equation for an electron in an electromagnetic field:

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \left[\frac{1}{2m}\left(i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2 + e\,\varphi\right]\Psi(x,t),$$

d) The electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are related to the electromagnetic potentials through.

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} - \nabla\varphi.$$

$\mathbf{E}$ and $\mathbf{B}$ are invariant under a gauge transformation

$$\mathbf{A} \longrightarrow \mathbf{A} + \nabla\chi, \qquad \varphi \longrightarrow \varphi - \frac{1}{c}\frac{\partial}{\partial t}\chi.$$

Show that Schrödinger's equation for a particle in an electromagnetic field is also invariant under gauge transformations provided that ψ transforms as

$$\psi \longrightarrow \exp\left(-\frac{ie}{\hbar c}\chi\right)\psi.$$

Exercise 4.2 Landau problem

Consider a magnetic field of strength B homogen in space and an electron moving in a plane perpendicular to it. We have tried in exercise 2.2 to compute naively the energy levels of this system, and we will now perform the same computation exactly in terms of quantum mechanics.

a) Check that $\phi = 0$ and $\mathbf{A} = \frac{1}{2}B\,(-y, x, 0)$ yields a constant magnetic field along the z direction. Write the Schrödinger equation with this ansatz in terms of (x, y) and their canonical conjugate momenta (p_x, p_y).

b) Let's define a new set of variables

$$\begin{array}{rclcrcl} q & = & a\,x + b\,p_y & \qquad & p & = & e\,y + f\,p_x \\ Q & = & c\,y + d\,p_x & \qquad & P & = & g\,x + h\,p_y \end{array},$$

where $a, \ldots, h$ are real coefficients. Find a set of coefficients such that the canonical commutation relations $[q,p] = [Q,p] = i\hbar$, $[q,P] = [Q,p] = [q,Q] = [p,P] = 0$ are satisfied and such that the terms proportional to qP and Qp in the Schrödinger equation vanish. *Hint: there are 6 equations for 8 free parameters, so one can choose for example* $a = c = 1/\sqrt{2}$.

What does the equation look like? What is the physical interpretation of the two sets (q,p) and (Q,P)?

c) Compute the energy levels and compare to the naive result of exercise 2.2.

Exercise 4.3 Particle in a one-dimensional square potential

The Schrödinger equation of a particle of mass m moving in a one-dimensional potential $V(x)$ is

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,t).$$

Since the right-hand side operator does not explicitly depend upon time, the solutions of this equation are stationary states of the form

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar},$$

where $\psi(x)$ is a solution of the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x).$$

A square potential is a potential $V(x)$ which remains constant everywhere along x except in a finite number of points where it has a discontinuity. $\psi''(x)$ has then also discontinuities, but $\psi'(x)$ and $\psi(x)$ are continuous everywhere. In a square potential the Schrödinger equation can be solved exactly in each region where the potential is constant, with the additional constraint that $\psi(x)$ and $\psi'(x)$ have to be continuous at each point of discontinuity of the potential.
Consider the potential

$$V(x) = \begin{cases} \infty & \text{for } x < 0, \\ V_1 & \text{for } 0 < x < a, \\ V_2 & \text{for } a < x < b, \\ 0 & \text{for } x > b, \end{cases}$$

where $V_1 < 0 < V_2$.

a) Investigate the motion of a particle moving in this potential in every energy range.

b) Compare the motion of a wave packet undergoing reflection at $x = 0$ with that of the corresponding classical particle.

c) Show the existence of resonances, i.e. values of the energy at which the intensity of the wave in the region $0 < x < a$ is of order one.

Exercise 4.4 Symplectic transformations

Consider a system described by n spatial operators q_i and momenta p_i obeying the canonical commutation relations: $[q_i, p_j] = i\hbar\ \delta_{ij}$, $[q_i, q_j] = [p_i, p_j] = 0$.
A linear symplectic transformation is a linear transformation taking the q_i, p_i into Q_i, P_i defined as

$$\begin{aligned} Q_i &= \sum_j A_{ij}\ q_j + \sum_j B_{ij}\ p_j \\ P_i &= \sum_j C_{ij}\ q_j + \sum_j D_{ij}\ p_j \end{aligned}$$

where the $n \times n$ matrices A, B, C and D can be arranged in a $2n \times 2n$ matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and M is symplectic, i.e. it obeys

$$M\Omega M^T = \Omega, \qquad \text{where} \quad \Omega = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}.$$

Show that any linear symplectic transformation is a symmetry of the quantum theory, namely that it preserves the commutation relations: $[Q_i, P_j] = i\hbar\ \delta_{ij}$, $[Q_i, Q_j] = [P_i, P_j] = 0$.