
Exercise 10.1 Time evolution of a density operator

We consider a spin-$\frac{1}{2}$ particle moving in an homogeneous magnetic field $\vec{B} = B\ \vec{n}$ ($|\vec{n}| = 1$). Forgetting about the spatial degrees of freedom, the Hamiltonian operator is then

$$H = \mu\ \vec{B} \cdot \vec{S}, \tag{1}$$

where $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ is the spin operator defined on the Hilbert space $\mathcal{H} = \mathbb{C}^2$ as $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, and $\vec{\sigma}$ is the vector of Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

In this exercise, we want to compute the time evolution of a density operator ρ_0. Since the Pauli matrices together with the identity form a basis for the 2×2 Hermitian matrices, ρ_0 can be written as

$$\rho_0 = \frac{1}{2}\left(\mathbb{1} + \vec{a} \cdot \vec{\sigma}\right), \qquad |a| \leq 1. \tag{3}$$

a) We have checked in exercise 8.3 that $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$ and $[\sigma_i, \sigma_j] = 2\epsilon_{ijk}\sigma_k$. Use these properties to show that $(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = (\vec{a} \cdot \vec{b})\mathbb{1} + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}$, and in particular $(\vec{n} \cdot \vec{\sigma})^2 = \mathbb{1}$.

b) Using the Taylor expansion for the time-evolution operator

$$U(t) := \exp\left(-itH/\hbar\right) = \sum_{k=0}^{\infty} \frac{\left(-itH/\hbar\right)^k}{k!}, \tag{4}$$

show that it can be written

$$U(t) = \cos\left(t\frac{\mu B}{2}\right)\mathbb{1} - i\sin\left(t\frac{\mu B}{2}\right)\vec{n} \cdot \vec{\sigma}. \tag{5}$$

Compute $U(t)^{-1}$.

c) Use these results to compute $\rho(t) = U(t)\rho_0 U(t)^{-1}$.
Show that if $\vec{a} \parallel \vec{B}$, $\rho(t)$ is constant in time.

Exercise 10.2 Combination of two spins $\frac{1}{2}$

Consider two systems described by 2-dimensional Hilbert spaces $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$ and two spin operators $\vec{S}_1$, $\vec{S}_2$ acting respectively on those two Hilbert spaces, and represented by $\vec{S}_i = \frac{\hbar}{2}\vec{\sigma}^{(i)}$, where $\vec{\sigma}^{(i)}$ is again the vector of Pauli matrices acting on $\mathcal{H}_i$.
We want to construct the eigenstates of the total spin operator $S = S_1 + S_2$ in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

a) Using the relation $\vec{S} = (\vec{S}_1 \otimes \mathbb{1}) + (\mathbb{1} \otimes \vec{S}_2)$, write explicitly the three components of $\vec{S}$ as 4×4 matrices and check that the relation $[S^i, S^j] = i\hbar\ \epsilon^{ijk} S^k$ holds. Compute S^2.

b) Check that the elements of the basis $\mathcal{B}_1 = \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ are all eigenvectors of S^z, but not of S^2.

   Find a set of normalized eigenvectors of both S^z and S^2.

c) Let $|L,l\rangle$ describe a state such that $S^2|L,l\rangle = \hbar^2\ L(L+1)\ |L,l\rangle$ and $S^z|L,l\rangle = \hbar\ l\ |L,l\rangle$.

   Show that $\mathcal{B}_2 = \{|1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle\}$ forms a basis of the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ and give an explicit definition of the elements of $\mathcal{B}_2$ in the basis $\mathcal{B}_1$.

   Check that under parity transformation (i.e. exchange of the two systems), the state $|L,l\rangle$ transforms as $P|L,l\rangle = (-1)^{L+1}|L,l\rangle$.

d) Let's define $S^\pm = \frac{1}{\sqrt{2}}(S^x \pm iS^y)$. How do the states in $\mathcal{B}_2$ transform under the action of $S^\pm$?

Exercise 10.3 The electron g-factor

The magnetic moment of a particle of charge q and mass m is related to its spin S through

$$\mu = g\frac{q}{2m}S \tag{6}$$

where g is a dimensionless constant called the g-factor.

a) Determine classically the g factor of the electron. In order to do so, let's assume that the electron is a sphere of radius R, of homogeneous mass and charge density, and rotating at an angular velocity ω. Its spin is then equal to its angular momentum $L = I\omega$, where I is the moment of inertia of the sphere.

   Hint: Remember that a charge q moving along a circular path yields a magnetic moment $\vec{\mu} = \frac{1}{2}q\ \vec{r} \times \vec{v}$, where r is the position of the charge relative to the center of the orbit and v its velocity.

A quantum mechanical treatement of the problem yields however a different result, namely $g = 2$. We want to show that a simple experiment can determine whether the g-factor of the electron is exactly 2 or differs slightly.

b) Consider an electron storage ring, i.e. a device in which a beam of electron is kept running on a circular orbit at constant velocity (for simplicity, the storage ring can be simply modelised by a cavity in which a homogeneous magnetic field is present).

   Show that the spin of an electron orbiting in the ring is precessing around the axis of the magnetic field and compute the frequency of this precession (known as the *Larmor precession*).

c) Let's assume that electrons are injected in the storage ring with a definite polarisation. Show that if the g factor of the electron is exactly two, at a given point the electrons will always have the same polarisation, and that this is not the case if g differs from 2, even slightly.

The experimentally measured value of g is actually slightly larger than 2, in agreement with the prediction from quantum electrodynamics.