Quantenmechanik I
Lösungsblatt 9

HS 09
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Exercise 9.1 Minimum uncertainty wavefunction

Assuming $\langle x\rangle = \langle p\rangle = 0$, the uncertainties come

$$\delta p = \sqrt{(\psi, p^2\psi)} = ||p\psi||,$$
$$\delta x = \sqrt{(\psi, x^2\psi)} = ||x\psi||.$$

The Heisenberg uncertainty relation comes from

$$\begin{aligned}
|(\psi, [x,p]\psi)| &= |(\psi, xp\psi) - (\psi, px\psi)| \\
&= |(x\psi, p\psi) - (p\psi, x\psi)| \\
&= |(x\psi, p\psi) - (x\psi, p\psi)^*| \\
&= |2i\Im(x\psi, p\psi)| \\
&\leq 2|(\psi, xp\psi)| \\
&\leq 2||x\psi||||p\psi|| \Rightarrow \frac{\hbar}{2} \leq \delta x\delta p
\end{aligned}$$

where $\Im$ stands for the imaginary part. To calculate the wave function that minimises $\delta x\delta p$, we need these two inequalities to become equalities.
For the first one,

$$|2i\Im(x\psi, p\psi)| = 2|(\psi, xp\psi) \Rightarrow \Re(x\psi, p\psi) = 0$$

In the second case, the Schwarz inequality becomes an equality if and only if the terms $x\psi$ and $p\psi$ are linearly dependent, ie $p\psi = \lambda x\psi$.
From the first condition we get

$$\begin{aligned}
\Re(x\psi, p\psi) &= (x\psi, p\psi) + (x\psi, p\psi)^* \\
&= (x\psi, p\psi) + (p\psi, x\psi) \Leftrightarrow \\
\Leftrightarrow 0 &= (\psi, (xp + px)\psi) \\
&= \lambda(x\psi, x\psi) + \lambda^*(x\psi, x\psi) \Rightarrow \\
\Rightarrow 0 &= \lambda + \lambda^* \\
&= \Re(\lambda),
\end{aligned}$$

ie, $\lambda = i\alpha$ for some real α. We then have

$$\begin{aligned}
p\psi(x) &= i\alpha x\psi(x) \Leftrightarrow \\
\Leftrightarrow -i\hbar\frac{\partial}{\partial x}\psi(x) &= i\alpha x\psi(x) \Leftrightarrow \\
\Leftrightarrow \frac{\partial}{\partial x}\psi(x) &= -\frac{\alpha}{\hbar}x\psi(x),
\end{aligned}$$

which results in the Gaussian wave function

$$\psi = A\exp\left(-\frac{\alpha x^2}{2\hbar}\right)$$

.
We set $\alpha > 0$ so that the integral of $\psi^*\psi$ is finite.

Exercise 9.2 Symmetry and projective representations – time translations

Time translations are represented by unitary operators $U(t)$.

a) Using the associativity of the matrix product,

$$\begin{aligned}
[U(x)U(y)] \quad U(z) &= U(x) \quad [U(y)U(z)] \Leftrightarrow \\
\Leftrightarrow w(x,y)U(x+y)U(z) &= U(x)w(y,z)U(y+z) \Leftrightarrow \\
\Leftrightarrow w(x,y)w(x+y,z)U(x+y+z) &= w(y,z)w(x,y+z)U(x+y+z) \Rightarrow \\
\Rightarrow w(x,y)w(x+y,z) &= w(y,z)w(x,y+z)
\end{aligned}$$

Setting $y = 0$ we get

$$\begin{aligned} w(x,0)w(x,z) &= w(0,z)w(x,z) \Rightarrow \\ &\Rightarrow w(x,0) = w(0,z) \end{aligned}$$

b) We want $\tilde{w}(t_1,t_2) = 1$.

$$\begin{aligned} \tilde{U}(t_1)\tilde{U}(t_2) &= \phi(t_1)U(t_1)\phi(t_2)U(t_2) \\ \tilde{w}(t_1,t_2)\tilde{U}(t_1+t_2) &= \phi(t_1)\phi(t_2)U(t_1)U(t_2) \\ \tilde{w}(t_1,t_2)\phi(t_1+t_2)U(t_1+t_2) &= \phi(t_1)\phi(t_2)w(t_1,t_2)U(t_1+t_2) \\ \tilde{w}(t_1,t_2) &= \frac{\phi(t_1)\phi(t_2)}{\phi(t_1+t_2)}w(t_1,t_2) = 1. \end{aligned}$$

c) To show that $w(t,0) = 1 \Rightarrow \phi(0) = 1$ we do

$$\begin{aligned} w(t_1,t_2) &= \frac{\phi(t_1+t_2)}{\phi(t_1)\phi(t_2)} \\ w(t,0) &= \frac{\phi(t+0)}{\phi(t)\phi(0)} \\ 1 &= \frac{1}{\phi(0)}. \end{aligned}$$

Let's now assume that $\phi(t)$ satisfies the condition given and test $\phi(t)e^{i\alpha t}$,

$$\frac{\phi(t_1)e^{i\alpha t_1}\phi(t_2)e^{i\alpha t_2}}{\phi(t_1+t_2)e^{i\alpha(t_1+t_2)}}w(t_1,t_2) = \frac{\phi(t_1)\phi(t_2)e^{i\alpha(t_1+t_2)}}{\phi(t_1+t_2)e^{i\alpha(t_1+t_2)}}w(t_1,t_2) = \frac{\phi(t_1)\phi(t_2)}{\phi(t_1+t_2)}w(t_1,t_2) = 1$$

d) w differentiable, $\phi'(0) = 0$, $\phi(0) = 1$.

$$\begin{aligned} \left.\frac{\partial}{\partial y}w(x,y)\right|_{y=0} &= \left.\frac{\partial}{\partial y}\frac{\phi(x+y)}{\phi(x)\phi(y)}\right|_{y=0} \\ &= \left.\frac{1}{\phi(x)}\frac{\partial}{\partial y}\frac{\phi(x+y)}{\phi(y)}\right|_{y=0} \\ &= \left.\frac{1}{\phi(x)}\frac{\left[\frac{\partial}{\partial y}\phi(x+y)\right]\phi(y) - \left[\frac{\partial}{\partial y}\phi(y)\right]\phi(x+y)}{\phi(y)^2}\right|_{y=0} \\ &= \frac{1}{\phi(x)}\frac{\left[\frac{\partial}{\partial x}\phi(x)\right]1 - 0\phi(x+y)}{1^2} \\ &= \frac{\frac{\partial}{\partial x}\phi(x)}{\phi(x)} \quad = \frac{\partial}{\partial x}\ln\phi(x). \end{aligned}$$

e) We will see that the system

$$\begin{cases} \left.\frac{\partial}{\partial y}w(x,y)\right|_{y=0} = \frac{\partial}{\partial x}\ln\phi(x), \\ \phi(0) = 1. \end{cases}$$

always has a solution when w verifies the cocycle condition. We start by applying a derivative in order to z to both sides of that equation, when $z = 0$,

$$
\begin{aligned}
w(x,y)w(x+y,z) &= w(y,z)w(x,y+z) \\
\frac{\partial}{\partial z} w(x,y)w(x+y,z)\bigg|_{z=0} &= \frac{\partial}{\partial z} w(y,z)w(x,y+z)\bigg|_{z=0} \\
w(x,y)\ \frac{\partial}{\partial z} w(x+y,z)\bigg|_{z=0} &= w(y,0)\ \frac{\partial}{\partial z} w(x,y+z)\bigg|_{z=0} + w(x,y+0)\ \frac{\partial}{\partial z} w(y,z)\bigg|_{z=0} \\
w(x,y)\frac{\partial}{\partial y}\ln\phi(x+y) &= 1\frac{\partial}{\partial y} w(x,y) + w(x,y)\frac{\partial}{\partial y}\ln\phi(y) \\
w(x,y)\left[\frac{\partial}{\partial y}\ln\phi(x+y) - \frac{\partial}{\partial y}\ln\phi(y)\right] &= \frac{\partial}{\partial y} w(x,y) \\
\frac{\partial}{\partial y}\left[\ln\phi(x+y) - \ln\phi(y)\right] &= \frac{\frac{\partial}{\partial y} w(x,y)}{w(x,y)} \\
\frac{\partial}{\partial y}\ln\frac{\phi(x+y)}{\phi(y)} &= \frac{\partial}{\partial y}\ln w(x,y) \\
\frac{\partial}{\partial y}\ln\frac{\phi(x+y)}{\phi(x)\phi(y)} &= \frac{\partial}{\partial y}\ln w(x,y),
\end{aligned}
$$

which recovers the result from d) at $y = 0$.

Exercise 9.3 Space translations in the plane

a) We have $A = iaP_1$ and $B = ibP_2$, which gives us the commutator

$$[A, B] = [iaP_1, ibP_2] = (ia)(ib)[P_1, P_2] = (ia)(ib)(i\alpha\mathbb{1}) = -i\alpha ab\mathbb{1}.$$

Since the commutator between these two operators is given by a constant times the identity, all commutators of higher order in the Baker-Campbell-Hausdorff formula vanish. For instance, $[A[A, B]] = -i\alpha ab[A, \mathbb{1}] = 0$. We have therefore

$$
\begin{aligned}
e^A e^B &= \exp(A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]] + \dots) \\
e^{iaP_1} e^{ibP_2} &= \exp(iaP_1 + ibP_2 + \frac{1}{2}(-i\alpha ab\mathbb{1}) + \frac{1}{12}0 - \frac{1}{12}0 + 0) \\
&= e^{i(aP_1+bP_2)} e^{\frac{-i\alpha ab}{2}}.
\end{aligned}
$$

b) In this more general case we have $A = i(aP_1 + bP_2)$, $B = i(a'P_1 + b'P_2)$, and the commutator is given by

$$
\begin{aligned}
[A, B] &= [i(aP_1 + bP_2), i(a'P_1 + b'P_2)] \\
&= -aa'[P_1, P_1] - ab'[P_1, P_2] - ba'[P_2, P_1] - bb'[P_2, P_2] \\
&= i\alpha(a'b - ab')\mathbb{1}.
\end{aligned}
$$

Applying two translations consecutively we obtain

$$
\begin{aligned}
T(\vec{r})T(\vec{r}') &= e^{i\vec{r}\cdot\vec{P}} e^{i\vec{r}'\cdot\vec{P}} \\
&= e^{i(\vec{r}+\vec{r}')\cdot\vec{P}} e^{\frac{i\alpha(a'b-ab')}{2}} \\
&= T(\vec{r}+\vec{r}')e^{\frac{-i\alpha\vec{r}\times\vec{r}'}{2}},
\end{aligned}
$$

where we defined $\vec{r}\times\vec{r}' = ab' - a'b$ (just the z component of the vector product, with no direction assigned).

We would like to define a gauge transformation $\tilde{T} = e^{i\theta(\vec{r})} e^{i\vec{r}\cdot\vec{P}}$ such that $\tilde{T}(\vec{r}+\vec{r'}) = \tilde{T}(\vec{r})\tilde{T}(\vec{r'})$. This would imply

$$
\begin{aligned}
e^{i\theta(\vec{r}+\vec{r}')}T(\vec{r}+\vec{r'}) &= e^{i\theta(\vec{r})}T(\vec{r})e^{i\theta(\vec{r}')}(\vec{r'}) \\
e^{i\theta(\vec{r}+\vec{r}')}T(\vec{r}+\vec{r'}) &= e^{i\theta(\vec{r})}e^{i\theta(\vec{r}')}T(\vec{r}+\vec{r}')e^{\frac{-i\alpha\vec{r}\times\vec{r}'}{2}} \\
\theta(\vec{r}+\vec{r}') &= \theta(\vec{r}) + \theta(\vec{r}') - \frac{\alpha}{2}(\vec{r}\times\vec{r}'),
\end{aligned}
$$

but it is impossible to define a function of the sum of two vectors that takes into account the vector product between (for instance, the sum is commutative while the vector product is anticommutative).

Exercise 9.4 Unitary and antiunitary symmetries

a unitary operator U will act on the scalar product as $(U\phi, U\psi) = (\phi, \psi)$. On the other hand, an antiunitary operator A will act as $(A\phi, A\psi) = (\phi, \psi)^* = (\psi, \phi)$.

a) In the exercise sheet, we have $T^2 = 1$ because we were considering our new favourite framework: dealing with bosons. I won't use that here, so I'd cut it from the exercise sheet. I'll check with JF this morning anyway.

The time evolution of a state of a system ruled by the Hamiltonian H is given by

$$\psi(t_1) = e^{-i(t_1-t_0)H/\hbar}\psi(t_0)$$

. When the time interval $\delta t = t_1 - t_0$ is very small we can write (for simplicity let's say $t_0 = 0$ and $\psi(0) = \psi$)

$$\psi(\delta t) = \left(\mathbb{1} - \frac{iH}{\hbar}\delta t\right)\psi.$$

The time reversal operator acts as

$$Te^{-iH\delta t}\psi = e^{-iH(-\delta t)}\psi,$$

which for small δt becomes

$$\begin{aligned} T\left(\mathbb{1} - \frac{iH}{\hbar}\delta t\right)\psi &= \left(\mathbb{1} - \frac{iH}{\hbar}(-\delta t)\right)T\psi, \quad \forall\psi \Rightarrow \\ \Rightarrow -\frac{i\delta t}{\hbar}HT\psi &= T\frac{i\delta t}{\hbar}H\psi, \quad \forall\psi \Rightarrow \\ \Rightarrow -iHT\psi &= TiH\psi, \quad \forall\psi \end{aligned}$$

sol. 1

An useful characteristic of unitary and antiunitary operators that follows from the way they act on the inner product is how they act complex numbers,

$$\begin{aligned} Uz &= zU, \qquad U \text{ unitary;} \\ Az &= z^*A, \qquad A \text{ antiunitary.} \end{aligned}$$

Suppose that T were unitary. In that case we would have

$$\begin{aligned} -iHT\psi &= iTH\psi, \quad \forall\psi \Leftrightarrow \\ \Leftrightarrow HT\psi &= -TH\psi, \quad \forall\psi. \end{aligned}$$

Consider now ψ_n to be an eigenstate of H of energy E_n. The correspondent time-reversed state is $T\psi_n$, which would have energy

$$HT\psi_n = -TH\psi_n = -E_nT\psi_n.$$

This would imply that the energy spectrum of a time-reversed system would be the symmetric of that of the original system. This does not make sense physically – the energy of the states should remain constant under time reversal. Consider for instance the case of a free particle. Its energy spectrum ranges from 0 to $+\infty$, and negative energies make no sense here (good old emotional argument)). If we want to say that a system presents time-reversal symmetry, then the spectrum of H should remain constant under that transformation, which is achieved if T is antiunitary,

$$\begin{aligned} -iHT\psi &= TiH\psi, \quad \forall\psi \Leftrightarrow \\ \Leftrightarrow -iHT\psi &= -iTH\psi, \quad \forall\psi \Leftrightarrow \\ \Leftrightarrow HT\psi &= TH\psi, \quad \forall\psi. \end{aligned}$$

We may also say we have $HT\psi = iTiH\psi$ and that one requirement for time symmetry is that the expectation value of H is invariant under time reversal,

$$\begin{aligned}
(\psi, H\psi) &= (T\psi, HT\psi) \\
&= (T\psi, iTiH\psi) \\
&= i(T\psi, TiH\psi) \\
&= \begin{cases} i(\psi, iH\psi), & T \text{ unitary} \\ i(\psi, iH\psi)*, & T \text{ antiunitary} \end{cases} \\
&= \begin{cases} -(\psi, H\psi), & T \text{ unitary} \\ i[i(\psi, H\psi)]^*, & T \text{ antiunitary} \end{cases} \\
&= \begin{cases} -(\psi, H\psi), & T \text{ unitary} \\ i(-i)(H\psi, \psi), & T \text{ antiunitary} \end{cases} \\
&= \begin{cases} -(\psi, H\psi), & T \text{ unitary} \\ (\psi, H\psi), & T \text{ antiunitary} \end{cases},
\end{aligned}$$

so T has to be antiunitary.

b) In the exercise sheet, $\vec{x} = (X, Y, Z)$ is the operator that measures the position.

The parity or space-inversion operator acts as

$$P\vec{x}\psi = -\vec{x}P\psi.$$

A reasonable requirement for parity is that the expectation value of $\vec{x}$ of a space-inverted state must be symmetric to the one of the original state,

$$\begin{aligned}
(P\psi, \vec{x}P\psi) = -(\psi, \vec{x}\psi), \forall\psi \Leftrightarrow \\
\Leftrightarrow -(P\psi, P\vec{x}\psi) = -(\psi, \vec{x}\psi),
\end{aligned}$$

which implies that P is unitary.