Exercise 10.1 Time evolution of a density operator

a)

$$\sigma_i\sigma_j = \frac{1}{2}\left(\{\sigma_i,\sigma_j\} + [\sigma_i,\sigma_j]\right) = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k$$

$$\begin{aligned}\Rightarrow \quad (\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) &= a_i b_j \sigma_i\sigma_j \\ &= a_i b_j\left(\delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k\right) \\ &= a_i b_i \mathbb{1} + i\epsilon_{ijk}\; a_i b_j \sigma_k = (\vec{a}\cdot\vec{b})\mathbb{1} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}\end{aligned}$$

b) Using the definition of H,

$$U(t) = \sum_{k=0}^{\infty}\frac{(-itH/\hbar)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-it\mu B/2)^k\,(\vec{n}\cdot\vec{\sigma})^k}{k!}$$

From (a) we have $(\vec{n}\cdot\vec{\sigma})^2 = \mathbb{1}$ and thus

$$(\vec{n}\cdot\vec{\sigma})^k = \begin{cases}\mathbb{1} & \text{for } k \text{ even}\\ \vec{n}\cdot\vec{\sigma} & \text{for } k \text{ odd}\end{cases}$$

Let's split the sum into even and odd k

$$\begin{aligned}U(t) &= \sum_{k=0}^{\infty}\frac{(-it\mu B/2)^{2k}\,(\vec{n}\cdot\vec{\sigma})^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{(-it\mu B/2)^{2k+1}\,(\vec{n}\cdot\vec{\sigma})^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k\,(t\mu B/2)^{2k}}{(2k)!} - i\sum_{k=0}^{\infty}\frac{(-1)^k\,(t\mu B/2)^{2k+1}}{(2k+1)!}\;\vec{n}\cdot\vec{\sigma} \\ &= \cos\left(t\frac{\mu B}{2}\right)\mathbb{1} - i\sin\left(t\frac{\mu B}{2}\right)\vec{n}\cdot\vec{\sigma}\end{aligned}$$

where in the last step we have recognized the Taylor expansion of the sinus and cosinus functions.

Finally

$$U(t)^{-1} = \exp(itH) = U(-t) = \cos\left(t\frac{\mu B}{2}\right)\mathbb{1} + i\sin\left(t\frac{\mu B}{2}\right)\vec{n}\cdot\vec{\sigma}$$

c) Writing $\chi = t\frac{\mu B}{2}$,

$$\begin{aligned}\rho(t) &= (\cos\chi\;\mathbb{1} - i\sin\chi\;\vec{n}\cdot\vec{\sigma})\,\frac{1}{2}\,(\mathbb{1} + \vec{a}\cdot\vec{\sigma})\,(\cos\chi\;\mathbb{1} + i\sin\chi\;\vec{n}\cdot\vec{\sigma}) \\ &= \frac{1}{2}\,(\cos\chi\;\mathbb{1} - i\sin\chi\;\vec{n}\cdot\vec{\sigma}) \\ &\quad\cdot\left[(\cos\chi + i\sin\chi\;\vec{a}\cdot\vec{n})\,\mathbb{1} + (\cos\chi\;\vec{a} + i\sin\chi\;\vec{n} - \sin\chi\;\vec{a}\times\vec{n})\cdot\vec{\sigma}\right]\end{aligned}$$

$$
\begin{aligned}
= \;& \frac{1}{2}\Big[\big(\cos^2\chi + i\sin\chi\cos\chi\;\vec{a}\cdot\vec{n} - i\sin\chi\cos\chi\;\vec{n}\cdot\vec{a} + \sin^2\chi\;\vec{n}^2 + i\sin^2\chi\;\vec{n}\cdot(\vec{a}\times\vec{n})\big)\,\mathbb{1} \\
& + \big(\cos^2\chi\;\vec{a} + i\sin\chi\cos\chi\;\vec{n} - \sin\chi\cos\chi\;\vec{a}\times\vec{n} - i\sin\chi\cos\chi\;\vec{n} + \sin^2\chi\;(\vec{a}\cdot\vec{n})\vec{n} \\
& \quad + \sin\chi\cos\chi\;\vec{n}\times\vec{a} + i\sin^2\chi\;\vec{n}\times\vec{n} - \sin^2\chi\;\vec{n}\times(\vec{a}\times\vec{n})\big)\cdot\vec{\sigma}\Big] \\
= \;& \frac{1}{2}\Big[\mathbb{1} + \big(\cos 2\chi\;\vec{a} + 2\sin^2\chi\;(\vec{a}\cdot\vec{n})\vec{n} - \sin 2\chi\;\vec{a}\times\vec{n}\big)\cdot\vec{\sigma}\Big]
\end{aligned}
$$

In the last step we have used $\vec{n}\cdot(\vec{a}\times\vec{n}) = 0$ and $\vec{n}\times(\vec{a}\times\vec{n}) = \vec{a} - (\vec{a}\cdot\vec{n})\vec{n}$.
If $\vec{a} \parallel \vec{n}$, we have $(\vec{a}\cdot\vec{n})\vec{n} = \pm|\vec{a}|\vec{n} = \vec{a}$ and $\vec{a}\times\vec{n} = 0$, so indeed

$$\rho(t) = \frac{1}{2}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right) = \rho_0$$

is time-independent.

Exercise 10.2 Combination of two spins $\frac{1}{2}$

a) By definition of the direct product of two matrices,

$$S_1^x \otimes \mathbb{1} = \frac{\hbar}{2}\sigma_1 \otimes \mathbb{1} = \frac{\hbar}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \mathbb{1}\otimes S_2^x = \frac{\hbar}{2}\mathbb{1}\otimes\sigma_1 = \frac{\hbar}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

$$\Rightarrow \quad S^x = (S_1^x \otimes \mathbb{1}) + (\mathbb{1}\otimes S_2^x) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Similarly, one finds

$$S^y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix} \qquad S^z = \hbar\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The relation $[S^i, S^j] = i\hbar\,\epsilon^{ijk}S^k$ can be verified by direct computation.
We have also

$$S^2 = \hbar^2\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

b) In this formalism, $|\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $|\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $|\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $|\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$,

so we see directly that they are all eigenvectors of S^z:

$$S^z|\uparrow\uparrow\rangle = \hbar|\uparrow\uparrow\rangle \qquad S^z|\downarrow\downarrow\rangle = -\hbar|\downarrow\downarrow\rangle \qquad S^z|\uparrow\downarrow\rangle = S^z|\downarrow\uparrow\rangle = 0$$

but not of S^2

$$S^2|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle \qquad S^2|\downarrow\downarrow\rangle = 2\hbar^2|\downarrow\downarrow\rangle \qquad S^2|\uparrow\downarrow\rangle = S^2|\downarrow\uparrow\rangle = \hbar^2\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

Actually $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ are already eigenvectors of S^2. For the others, it is straightforward to find normalised linear combinations that are indeed eigenvectors:

$$S^2\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) = 2\hbar^2\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \qquad S^2\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = 0$$

c) S^z and S^2 are Hermitian, therefore we have

$$\begin{aligned}
\langle L_1, l_1|S^z|L_2, l_2\rangle = \hbar l_1 \langle L_1, l_1|L_2, l_2\rangle = \hbar l_2 \langle L_1, l_1|L_2, l_2\rangle \quad &\Rightarrow \quad \langle L_1, l_1|L_2, l_2\rangle = 0 \text{ for } l_1 \neq l_2 \\
\langle L_1, l_1|S^2|L_2, l_2\rangle = \hbar^2 L_1(L_1+1)\langle L_1, l_1|L_2, l_2\rangle & \\
= \hbar^2 L_2(L_2+1)\langle L_1, l_1|L_2, l_2\rangle \quad &\Rightarrow \quad \langle L_1, l_1|L_2, l_2\rangle = 0 \text{ for } L_1 \neq L_2
\end{aligned}$$

Therefore all the four vectors in $\mathcal{B}_2$ are orthogonal to each other, thus linearly independent. Hence if they exist and are non-zero, they form automatically a basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

This is indeed the case: from the point (b), we can take

$$|1,1\rangle = |\uparrow\uparrow\rangle \qquad |1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \qquad |1,-1\rangle = |\downarrow\downarrow\rangle \qquad |0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

and they satisfy the conditions $S^2|L,l\rangle = \hbar^2\ L(L+1)\ |L,l\rangle$ and $S^z|L,l\rangle = \hbar\ l\ |L,l\rangle$ (in any other choice of the basis $\mathcal{B}_2$, the basis vector are just real multiples of these, so we can use this choice as the most general one).

Using the definition above, we see directly that only $|0,0\rangle$ is antisymmetric under spin exchange, the others being symmetric. Hence $P|L,l\rangle = (-1)^{L+1}|L,l\rangle$ is satisfied.

d) We have $|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $|1,0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $|1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $|0,0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$,

and by definition

$$S^+ = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad S^- = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Hence

$$\begin{array}{llll}
S^+|1,1\rangle = 0 & S^+|1,0\rangle = \frac{\hbar}{2}|1,1\rangle & S^+|1,-1\rangle = \frac{\hbar}{2}|1,0\rangle & S^+|0,0\rangle = 0 \\
S^-|1,1\rangle = \frac{\hbar}{2}|1,0\rangle & S^-|1,0\rangle = \frac{\hbar}{2}|1,-1\rangle & S^-|1,-1\rangle = 0 & S^-|0,0\rangle = 0
\end{array}$$

The operators $S^\pm$ are therefore raising and lowering the index l, but only in the domain $|l| \leq L$.

Exercise 10.3 The electron g-factor

a) We consider a sphere of radius R, with homogeneous charge density ρ, rotating along the z axis with angular velocity ω.

Each infinitesimal piece of this sphere carries a charge dq and induces an infinitesimal magnetic momentum along the z axis

$$d\mu = \frac{1}{2}dq\ rv = \frac{1}{2}dq\ r^2\omega = \frac{1}{2}\rho\omega r^2\ dV$$

so that the total magnetic moment of the electron is (subsituting $\rho = \frac{e}{V}$)

$$\mu = \frac{e}{2V}\omega\int r^2\ dV = \frac{e}{2m}\omega\int r^2\ dm$$

In the last equality we have used the fact that the mass density is also homogeneous. The integral is nothing but the moment of inertia I of the sphere, and ωI is then the angular momentum L (or equivalently the spin S):

$$\mu = \frac{eS}{2m}$$

Hence $g = 1$.

b) Let the homogeneous magnetic field $\vec{B}$ be along the z axis: $\vec{B} = B\hat{e}_z$. From exercise 2.2, one can see that the electrons orbit in the storage ring at a frequency $\omega_c = eB/m$, which is known as the cyclotron frequency.

The magnetic field exerts a torque $\vec{\tau}$ on the magnetic moment of the electron:

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

and according to the definition

$$\vec{\mu} = g\frac{e}{2m}I\vec{\Omega}$$

where $\vec{\Omega}$ is the rotation frequency of the electron and I its moment of inertia.

The equivalent of Newton's second law for rotations reads

$$\vec{\tau} = I\dot{\vec{\Omega}}$$

which yields

$$\dot{\vec{\Omega}} = g\frac{e}{2m}\vec{\Omega} \times \vec{B} = \frac{g}{2}\omega_c\, \vec{\Omega} \times \hat{e}_z$$

$\vec{\Omega}$ is time dependent. One can parametrise it as

$$\vec{\Omega} = \Omega(t) \begin{pmatrix} \sin\theta(t)\sin\phi(t) \\ \sin\theta(t)\cos\phi(t) \\ \cos\theta(t) \end{pmatrix}$$

so that the equation above becomes

$$\dot{\Omega} \begin{pmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{pmatrix} + \Omega\dot{\theta} \begin{pmatrix} \cos\theta\sin\phi \\ \cos\theta\cos\phi \\ -\sin\theta \end{pmatrix} + \Omega\dot{\phi} \begin{pmatrix} \sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ 0 \end{pmatrix} = \frac{g}{2}\omega_c\Omega \begin{pmatrix} \sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ 0 \end{pmatrix}$$

The solution is then obviously $\dot{\Omega} = \dot{\theta} = 0$ and $\dot{\phi} = \frac{g}{2}\omega_c$. This means that the magnetic momentum $\vec{\mu}$ is precessing around the axis z with an angular velocity $\omega_p = \frac{g}{2}\omega_c$.

c) The polarisation remains the same afer any number of turns if and only if the precession of the electrons' magnetic moment has the same frequency as the rotation of the electrons in the ring (or an integer multiple thereof), i.e. if $\omega_p = k\ \omega_c$, $k \in \mathbb{N}$.

From the result of point (b), it is straigthforward to see that this is the case if g equals exactly 2.