### Exercise 13.1 Dielectric Susceptibility of Free Electrons

Consider a non-interacting one-dimensional gas of spinless electrons,

$$\mathcal{H} = \frac{1}{L}\sum_k \frac{k^2}{2m} c_k^\dagger c_k \qquad \text{with } \hbar = 1 \; . \tag{1}$$

We want to evaluate its linear response to an external scalar potential, i.e. a perturbation of the form

$$\delta\mathcal{H} = -e\int dx\, \phi(x,t)\, \hat{n}(x) = -e\int dx\, \phi(x,t)\, c^\dagger(x) c(x) \; . \tag{2}$$

The density operator $\hat{n}(x)$ is defined by the second equality.

a) To compute the linear response, use the Kubo-formalism (section 6.1) for the dielectric susceptibility,

$$\chi_e(x-x', t-t') = -i\Theta(t-t')e^2 \left\langle \left[\hat{n}_H(x,t), \hat{n}_H(x',t')\right]\right\rangle_{\mathcal{H}} \; , \tag{3}$$

where $\hat{n}_H(x,t)$ represents the density operator in the Heisenberg picture. Show that the Fourier transform $\chi(q,\omega)$ of $\chi_e(x-x', t-t')$ can be written as

$$\chi(q,\omega) = e^2 \sum_k \frac{f(\epsilon_{k+q}) - f(\epsilon_k)}{\omega + \epsilon_k - \epsilon_{k+q} + i\eta} \; , \tag{4}$$

where $f(\epsilon)$ denotes the Fermi function.

b) The imaginary part of the so-called Lindhard function $\chi(q,\omega)$ obtained in a) encodes the spectrum of the (charge) excitations that couple to $\phi(x,t)$. Derive conditions for the region in the $(q,\omega)$-plane for which $\mathrm{Im}\,\chi(q,\omega) \neq 0$ holds and make a schematic graph. Argue that the "particle-hole excitations" also fulfill the same conditions.

*Hint*: In order to consider excitations from the ground state, consider the zero temperature limit and first show that $\chi(q,\omega)$ can be written as

$$\chi(q,\omega) = \sum_{|k|\le k_F} \left[\frac{1}{\omega + \epsilon_k - \epsilon_{k+q} + i\eta} - \frac{1}{\omega - \epsilon_k + \epsilon_{k+q} + i\eta}\right] \; . \tag{5}$$

Then, take the continuum limit and obtain $\mathrm{Im}\,\chi(q,\omega)$ using the Dirac identity

$$\frac{1}{x \pm i0} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x) \; , \tag{6}$$

where $\mathcal{P}$ denotes the Cauchy principal value, and integration over $x$ is implied.

**Office Hours:** Monday, December 14, 8-10 am (HIT K 12.2)