

Aufgabe 8.1 Ice Skating

The small friction of skates on ice is usually explained by a small film of water that forms between the blade and the ice's surface. In this exercise we would like to investigate whether the formation of this film can be explained by the pressure exerted by the skater on the ice. From the phase diagram of water, we know that applying pressure decreases the melting temperature, thus allowing the ice to melt below 0°C .

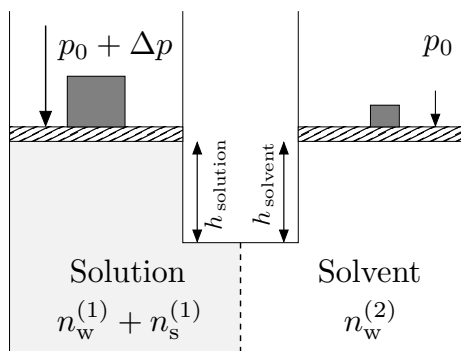
1. Assuming a skater of 80 kg with blades of 100 mm^2 , what is the resulting pressure?
2. By how many K is the melting temperature decreased by this pressure? From the lecture (script: eq. 7.40) we know that:

$$\frac{dp}{dT} = -138\text{ atm/K} \approx -138 \cdot 10^5 \frac{\text{N}}{\text{m}^2\text{K}}$$

3. Skating ice typically has a temperature of $T_{\text{ice}} \approx -5^\circ\text{C}$. Can this effect of lowering the melting point be used to (solely) explain the melting of the ice?

Aufgabe 8.2 Osmotic pressure

Consider the following experimental arrangement for the measurement of the osmotic pressure Δp at constant temperature T :



Two sections are separated by a semipermeable wall, through which only the solvent (i.e. water) can pass. In addition section (1) contains $n_s^{(1)}$ moles of solute (i.e. sugar). This causes a difference in the chemical potential of the two sides, which in turn would cause the solvent to flow from (2) to (1). In this exercise we want to calculate the additional pressure Δp (osmotic pressure) required to prevent this flow.

1. Express the Gibbs energy for both sides as a function of pressure p , temperature T and moles of solvent n_w and solute n_s in the respective section. Use $g_i(p, T)$ to represent the Gibbs energy per mole of each substance i (script: eq. 8.10).
2. Use the equilibrium requirement for the chemical potentials to relate the Gibbs potentials of the two sides. You may assume that $n_s^{(1)} \ll n_w^{(1)}$. From the resulting equation, you can find the osmotic pressure Δp (script: chapter 8.2).
3. What would happen if the additional pressure Δp was removed?

Aufgabe 8.3 Nucleation *

Consider a d -dimensional elastic membrane (elasticity C) in $d + 1$ dimensions, which can be deformed along the $d + 1$ th “transversal” direction (see figure). This transversal deformation can be described with $u(\vec{x})$, where $\vec{x} \in \mathbb{R}^d$. The deformation energy is $\propto (\nabla u)^2$ and the potential for the deformation is made up of a periodic part and of a coupling to a constant external force F :

$$V(u) = V_0 (1 - \cos(k_0 u)) - Fu.$$

Then the free energy for the membrane is given by

$$H[u] = \int d^d x \left[\frac{C}{2} (\nabla u)^2 + V(u) \right]$$

1. Draw the potential for different values of the external force F ? Which possibilities exist for the mobility of the membrane along the transverse direction? Define and determine a critical external force F_c .
2. Let's consider now $F \ll F_c$ and $d \geq 2$. Calculate the approximate energy $U(R)$ of a *nucleus* with radius R . This is a local deformation of the membrane, which disappears outside R and has the constant value $u = u_0 = \frac{2\pi}{k_0}$ inside the hypersphere of radius R . Make the assumption that the elastic and potential energy are of the same size for $F = 0$ and the deformation increases from 0 to u_0 in a small area of width w . Determine the critical radius R_c , above which the nucleus begins to grow. Why does $U(R_c)$ correspond to an activation energy, and how does it vary with F ?
3. With $V_0 = 0$ the membrane moves according to $v = F/\eta$, with a speed v and a friction coefficient η . Determine for our case with periodic potential and temperature $T > 0$, the drift velocity v as a function of the force F ; consider, in particular, the cases $F \rightarrow 0$ and $F \gg F_c$.
4. In one dimension (elastic line) $u_{\text{kink}}(x) = \frac{4}{k_0} \arctan(e^{x/w})$ with a constant w is a minimum of the above energy function (free energy equation) for $F = 0$ which can represent a flank of the nucleus. The energy for a nucleus with radius R and with such two flanks is then

$$U(R) \approx 2E_{\text{kink}}(1 - e^{-R/w}) - Fu_0 R.$$

Calculate the critical radius R_c and the activation energy. What is different if $F \rightarrow 0$ and what are the consequences for the speed of the line?

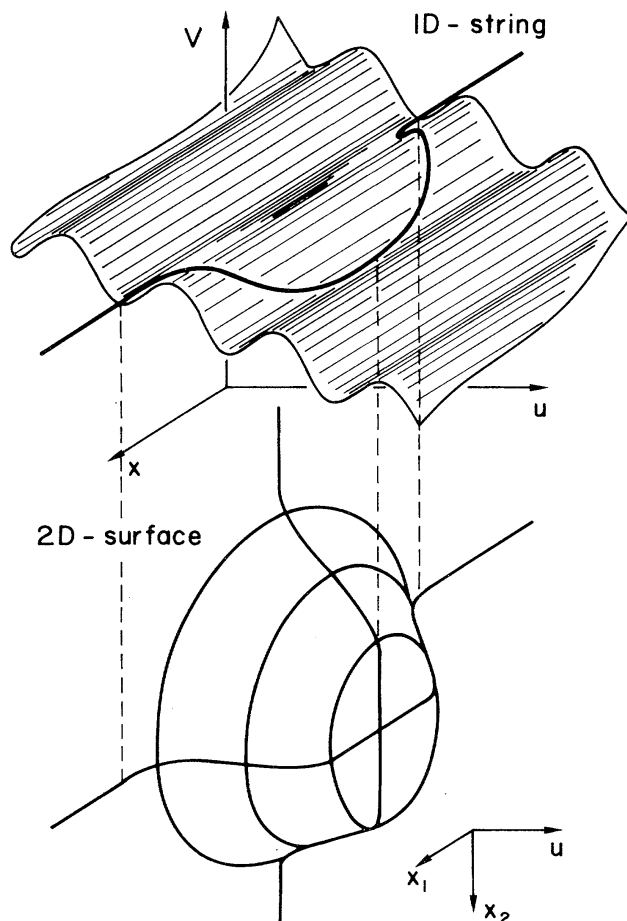


FIG. 13. Elastic manifold trapped in a (tilted) washboard potential. Top: One-dimensional elastic string with a finite segment (nucleus) activated to the next valley. The activation energy $2E_k$ involves the production of two kinks and remains always finite—the string is never in a “glassy” state. Bottom: Two-dimensional elastic surface with a finite nucleus activated to the next valley. The activation energy involves the creation of a one-dimensional (thin) wall, which costs an energy $2\pi r E_k$, where r is the radius of the nucleus. If the nucleus is large enough, $r > r_c$, it expands and the elastic manifold moves on to the next valley. The critical radius r_c increases with decreasing driving force F , $r_c = E_k / u_0 F$, and the manifold shows glassy behavior with a diverging activation energy at vanishing driving force, $U(F) = \pi E_k^2 / u_0 F$.

More background on the task in: G. Blatter *et al.*, Rev. Mod. Phys. **66** 1125, (1994).