Chapter 5

Elements of quantum electrodynamics

5.1 Quantum mechanical equations of motion

In quantum mechanics I & II, the correspondence principle played a central role. It is in a sense the recipe to quantize a system whose Hamiltonian is known. It consists in the following two substitution rules :

$$\vec{p} \longmapsto -i\nabla \qquad \text{(momentum)}, \tag{5.1}$$

$$E \longmapsto i\partial_t \qquad \text{(energy)}. \tag{5.2}$$

For nonrelativistic quantum mechanics we get the celebrated **Schrödinger equation**,

$$i\partial_t\psi = H\psi, \qquad \text{with } H = -\frac{1}{2m}\triangle + V(\vec{x}), \tag{5.3}$$

whose free solution ($V(\vec{x}) \equiv 0$) is,

$$\psi(\vec{x}, t) = C\mathrm{e}^{i(Et-\vec{p}\cdot\vec{x})} \qquad \text{with } E = \frac{\vec{p}^{\,2}}{2m}.$$

The relativistic version of the energy-momentum relationship is however,

$$E^2 = \vec{p}^{\,2} + m^2, \tag{5.4}$$

from which we get, using again the correspondence principle Eq. (5.1),

$$-\partial_t^2\psi = (-\triangle + m^2)\psi. \tag{5.5}$$

At this point we define some important symbols which will follow us troughout the rest of this lecture,

$$\partial_\mu := (\partial_t, \nabla), \tag{5.6}$$
$$\partial^\mu := g^{\mu\nu}\partial_\nu = (\partial_t, -\nabla), \tag{5.7}$$
$$\Box := \partial_\mu \partial^\mu = \partial_t^2 - \triangle. \tag{5.8}$$

With this notation we can then reformulate Eq. (5.5) to get the **Klein-Gordon equation**,

$$\boxed{\left(\Box + m^2\right)\psi = 0}, \tag{5.9}$$

with solutions,

$$\psi(\vec{x}, t) = C\mathrm{e}^{i(Et - \vec{p}\cdot\vec{x})} \qquad \text{with } E = \pm\sqrt{\vec{p}^{\,2} + m^2}.$$

We see that in this case it is possible to have negative energy eigenvalues, a fact not arising with the nonrelativistic case.

As in the case of the Schrödinger equation (5.3) we can formulate a continuity equation. To do so we multiply the Klein-Gordon equation (5.9) by the left with ψ^* and its conjugate, $(\Box + m^2)\,\psi^* = 0$, by ψ and then subtract both equations to get,

$$\begin{aligned} 0 &= \psi^*\partial^\mu\partial_\mu\psi - \psi\partial^\mu\partial_\mu\psi^* \\ &= \partial^\mu(\psi^*\partial_\mu\psi - \psi\partial_\mu\psi^*) \\ &\Rightarrow \partial_t(\psi^*\partial_t\psi - \psi\partial_t\psi^*) + \nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*) = 0. \end{aligned} \tag{5.10}$$

We would like to interpret $\psi^*\partial_t\psi - \psi\partial_t\psi^*$ in Eq. (5.10) as a probability density, or more exactly,

$$\rho = i(\psi^*\partial_t\psi - \psi\partial_t\psi^*),$$

which is *not* a positive definite quantity (as we can convince ourselves by computing ρ for the plane wave solution), and hence cannot be interpreted as a probability density as it was the case in QM.

When computing the continuity equation for the Schrödinger equation, where such a problem does not arise, we see that the problem lies essentially in the presence of a *second* order time derivative in the Klein-Gordon equation.

We now make a big step, by imposing that our equation of motion only contains a *first* order time derivative. Since we want a Lorentz invariant equation of motion, we conclude that only a *linear* dependence on ∇ is allowed. Following Dirac's intuition, we make the ansatz,

$$(i\gamma^\mu\partial_\mu - m)\psi = (i\gamma^0\partial_t + i\vec{\gamma}\cdot\nabla - m)\psi = 0. \tag{5.11}$$

Turning back to the correspondence principle we remark that,

$$(\gamma^0 E - \vec{\gamma}\cdot\vec{p} - m)\psi = 0 \qquad \Rightarrow (\gamma^0 E - \vec{\gamma}\cdot\vec{p} - m)^2\psi = 0, \tag{5.12}$$

which must stay compatible with the mass-shell relation, $E^2 = \vec{p}^2 + m^2$.

This implies that the γ^μ's cannot be numbers since it would then be impossible to satisfy,

$$(\vec{\gamma}\cdot\vec{p})^2 = \left(\sum_{i=1}^{3}\gamma^i p_i\right)^2 \overset{!}{\propto} \vec{p}^2,$$

so we let them be $n \times n$ matrices, for an n which is still to be determined.

We now derive relations that the γ^μ's must fullfill, so that the mass-shell relation remains true. From Eq. (5.12), and again with the correspondence principle, we must have,

$$\underbrace{i\partial_t}_{E} = (\gamma^0)^{-1}\vec{\gamma}\cdot\underbrace{(-i\nabla)}_{\vec{p}} + (\gamma^0)^{-1}m$$

$$\Rightarrow \underbrace{-\partial_t^2}_{E^2} = -\sum_{i,j=1}^{3}\frac{1}{2}\left((\gamma^0)^{-1}\gamma^i(\gamma^0)^{-1}\gamma^j + (\gamma^0)^{-1}\gamma^j(\gamma^0)^{-1}\gamma^i\right)\partial_i\partial_j \tag{5.13}$$

$$- i\cdot m\sum_{i=1}^{3}\left((\gamma^0)^{-1}\gamma^i(\gamma^0)^{-1} + (\gamma^0)^{-1}(\gamma^0)^{-1}\gamma^i\right)\partial_i \tag{5.14}$$

$$+ m^2(\gamma^0)^{-1}(\gamma^0)^{-1} \tag{5.15}$$

$$\overset{!}{=}(\partial_i\partial_i + m^2) \tag{5.16}$$

$$=(\underbrace{-\triangle}_{\vec{p}^2} + m^2).$$

Comparing Eqs. (5.16) and (5.15) we conclude that,

$$(\gamma^0)^{-1}(\gamma^0)^{-1} \overset{!}{=} \mathbb{1} \Rightarrow (\gamma^0)^{-1} = \gamma^0. \tag{5.17}$$

Defining $\{a, b\} := ab + ba$ and comparing Eq. (5.16) and Eq. (5.14) we get

$$\gamma^0\gamma^i(\gamma^0)^{-1} \overset{!}{=} 0 \Rightarrow \{\gamma^i, \gamma^0\} = 0. \tag{5.18}$$

Finally, comparing Eq. (5.16) and Eq. (5.13), we have,

$$-\frac{1}{2}\left((\gamma^0)^{-1}\gamma^i(\gamma^0)^{-1}\gamma^j + (\gamma^0)^{-1}\gamma^j(\gamma^0)^{-1}\gamma^i\right) \overset{!}{=} \delta_{ij} \Rightarrow \{\gamma^i, \gamma^j\} = -2\delta_{ij}. \tag{5.19}$$

We can summarize Eqs. (5.17), (5.18) and (5.19) in the **Clifford algebra of the γ-matrices**,

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}}, \tag{5.20}$$

where $g^{00} = 1$, $g^{ii} = -1$ and all the other elements vanish.

Important facts : The eigenvalues of γ^0 can only be ± 1 and those of γ^i $\pm i$ and the γ-matrices have vanishing trace :

$$\begin{aligned}
\operatorname{Tr}\gamma^i &= \operatorname{Tr}(\gamma^0\gamma^0\gamma^i) = -\operatorname{Tr}(\gamma^0\gamma^i\gamma^0) = -\operatorname{Tr}\gamma^i \Rightarrow \operatorname{Tr}\gamma^i = 0, \\
\operatorname{Tr}\gamma^0 &= \operatorname{Tr}(\gamma^0\gamma^i(\gamma^i)^{-1}) = -\operatorname{Tr}(\gamma^i\gamma^0(\gamma^i)^{-1}) = -\operatorname{Tr}(\gamma^0) \Rightarrow \operatorname{Tr}\gamma^0 = 0.
\end{aligned}$$

The eigenvalue property of γ^0 implies with the last equation that the dimension n of the γ-matrices must be even.

For $n = 2$ there are no matrices satisfying Eq. (5.20), as can be checked by direct computation.

For $n = 4$ there are many possibilities. The most common choice in textbooks is the Dirac-Pauli representation :

$$\gamma^0 = \mathbb{1} \otimes \sigma^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \sigma^i \otimes (i\sigma^2) = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{5.21}$$

with the Pauli matrices,

$$\sigma^0 = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the Kronecker product of 2×2-matrices,

$$A \otimes B = \begin{pmatrix} b_{11}A & b_{12}A \\ b_{21}A & b_{22}A \end{pmatrix}.$$

Looking at the **Dirac equation**

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0} \tag{5.22}$$

we see that ψ is no longer a function but a vector, called **(4-)spinor**,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}.$$

For 4-spinors, there are two types of adjoints, namely,

- the hermitian adjoint $\psi^\dagger = (\psi_1^* \, \psi_2^* \, \psi_3^*, \psi_4^*)$, and
- the Dirac adjoint $\bar{\psi} := \psi^\dagger \gamma^0 = (\psi_1^* \, \psi_2^* \, -\psi_3^*, -\psi_4^*)$.

Note that $\bar{\psi}$ satisfies a dirac equation of its own,

$$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0. \tag{5.23}$$

We now focus our attention on the continuity equation for the Dirac field. From Eqs. (5.22) and (5.23),

$$i\psi^\dagger(\partial_t \psi) = \left(-i\psi^\dagger \gamma^0 \gamma^i \partial_i + \psi^\dagger \gamma^0 m\right)\psi,$$

and its hermitian conjugate,

$$-i(\partial_t \psi^\dagger)\psi = \left(i(\partial_i \psi^\dagger)\gamma^0\gamma^i + \psi^\dagger\gamma^0 m\right)\psi,$$

we get the difference,

$$\begin{aligned} \partial_t(\psi^\dagger \psi) &= -\left[(\partial_i \psi^\dagger)\gamma^0\gamma^i\psi + \psi^\dagger\gamma^0\gamma^i i(\partial_i\psi)\right], \\ \partial_t(\bar{\psi}\gamma^0\psi) &= -\partial_i(\bar{\psi}\gamma^i\psi). \end{aligned} \tag{5.24}$$

We identify the components as,

$$\rho = \bar{\psi}\gamma^0\psi, \qquad \vec{j} = \bar{\psi}\vec{\gamma}\psi,$$

or interpreting them as components of a 4-vector as in classical electrodynamics,

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \tag{5.25}$$

we see that Eq. (5.24) can be reexpressed in the manifestly covariant form,

$$\boxed{\partial_\mu j^\mu = 0}. \tag{5.26}$$

5.2 Solutions of the Dirac equation

Before we look at the solutions of the free Dirac equation, we introduce the slash notation for contraction with the γ-matrices : $\not{a} := \gamma^\mu a_\mu$. The Dirac equation then reads $(i\not{\partial} - m)\psi = 0$.

5.2.1 Free particle at rest

In the rest frame of a particle, the Dirac equation reduces to,

$$i\gamma^0\partial_t\psi = m\psi,$$

for which we find four linearly indepedent solutions, namely,

$$\psi_1 = \mathrm{e}^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \psi_2 = \mathrm{e}^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad E = m \qquad \text{(particles)}$$
$$\psi_3 = \mathrm{e}^{+imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \psi_4 = \mathrm{e}^{+imt}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad E = -m \qquad \text{(antiparticles).}$$

5.2.2 Free particle

In order to preserve the Lorentz invariance of a solution, it must only depend on Lorentz scalars – quantities which are invariant under Lorentz transformations – like $p \cdot x = p_\mu x^\mu$. We make the ansatz,

$$\begin{aligned}\psi_{1,2} &= \mathrm{e}^{-ip\cdot x}u_\pm(p), \qquad p^0 > 0\\ \psi_{3,4} &= \mathrm{e}^{+ip\cdot x}v_\mp(-p), \qquad p^0 < 0.\end{aligned}$$

Pluging those ansatz in the Dirac equation, we get,

$$(\not{p} - m)u_\pm(p) = \bar{u}_\pm(\not{p} - m) = 0, \tag{5.27}$$
$$(\not{p} + m)v_\pm(p) = \bar{v}_\mp(\not{p} + m) = 0, \tag{5.28}$$

where we replaced $-p \mapsto p$ in the second equation, having thus $p^0 > 0$ in both cases now.

5.2.3 Explicit form of u and v

As checked in the exercices, the explicit form for the u and v functions are,

$$u_\pm(p) = \sqrt{p^0 + m}\begin{pmatrix}\chi_\pm\\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0+m}\chi_\pm\end{pmatrix}, \tag{5.29}$$
$$v_\pm(p) = \sqrt{p^0 + m}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{p^0+m}\chi_\mp\\ \chi_\mp\end{pmatrix}, \tag{5.30}$$

where $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to a "spin up" state and $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to a "spin down" state.

We note on the way that the application,

$$\vec{p} \longmapsto \vec{\sigma} \cdot \vec{p} = \sigma^i p_i = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix},$$

defines an isomorphism between the vector spaces of 3-vectors and hermitian 2×2-matrices.

5.2.4 Operators on spinor spaces

Hamiltonian The Hamiltonian is defined by $i\partial_t \psi = H\psi$. Isolating the time derivative in the Dirac equation, Eq. (5.22), we read out,

$$H = -i\gamma^0\gamma^i\partial_i + \gamma^0 m = \begin{pmatrix} m\mathbb{1} & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m\mathbb{1} \end{pmatrix}. \tag{5.31}$$

Helicity The helicity is the compenent of the spin in the direction of motion $\hat{\vec{p}} := \frac{\vec{p}}{|\vec{p}|}$, and is defined by,

$$h = \frac{1}{2}\vec{\sigma} \cdot \hat{\vec{p}} \otimes \mathbb{1} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} \cdot \hat{\vec{p}} & 0 \\ 0 & \vec{\sigma} \cdot \hat{\vec{p}} \end{pmatrix}. \tag{5.32}$$

By direct computation, one can check that $[H, h] = 0$, and thus there exist a set of eigenfunctions diagonalizing H and h simultaneously. The eigenvalues of h are then constants of the motion and hence good quantum numbers to label the corresponding states.

This quantum number λ can take two values,

$$\lambda = \begin{cases} +\frac{1}{2} & \text{positive helicity} \leftrightsquigarrow \vec{s} \upharpoonleft\upharpoonright \vec{p}, \\ -\frac{1}{2} & \text{negative helicity} \leftrightsquigarrow \vec{s} \upharpoonleft\downharpoonright \vec{p}. \end{cases} \tag{5.33}$$

We stress here that helicity/handedness is *not* a Lorentz invariant quantity for massive particles.

Consider $\vec{p}$ in the z-direction, then,

$$\frac{1}{2}\vec{\sigma} \cdot \hat{\vec{p}}\chi_\pm = \frac{1}{2}\sigma^3\chi_\pm = \pm\frac{1}{2}\chi_\pm.$$

From the last argumentative steps, we are not surprised with the statement that the Dirac equation describes spin-$\frac{1}{2}$ particles.

Chirality Consider the Dirac equation for the case of massless particles. This is a good approximation for $E \gg m$, which is often the case in accelerator experiments. Setting $m = 0$ simplifies Eq. (5.22) leading to

$$i\gamma^\mu \partial_\mu \psi = 0.$$

Eq. (5.29) and (5.30) change accordingly. We consider for now the particle solutions $u_\pm$:

$$u_\pm(p) = \sqrt{|\vec{p}|} \begin{pmatrix} \chi_\pm \\ \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi_\pm \end{pmatrix} = \sqrt{|\vec{p}|} \begin{pmatrix} \chi_\pm \\ \pm\chi_\pm \end{pmatrix}. \tag{5.34}$$

It is convenient to define the so-called chirality matrix γ_5:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

which in the Dirac-Pauli representation reads

$$\gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Using that the γ-matrices fulfill $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ (see Eq. (5.20)), one can show that

$$\{\gamma_5, \gamma^\mu\} = 0 \text{ and} \tag{5.35}$$

$$\gamma_5^2 = \mathbb{1}. \tag{5.36}$$

These properties of γ_5 imply that if ψ is a solution of the Dirac equation then so is $\gamma_5\psi$. Furthermore, since $\gamma_5^2 = \mathbb{1}$ the eigenvalues of the chirality matrix are ± 1:

$$\gamma_5\psi_\pm = \pm\psi_\pm$$

which defines the chirality basis $\psi_\pm$.

Let us apply the γ_5 matrix to the spinor part of particle solutions of the free Dirac equation given in Eq. (5.34):

$$\gamma_5 u_\pm(p) = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \sqrt{|\vec{p}|} \begin{pmatrix} \chi_\pm \\ \pm\chi_\pm \end{pmatrix} = \sqrt{|\vec{p}|} \begin{pmatrix} \pm\chi_\pm \\ \chi_\pm \end{pmatrix} \tag{5.37}$$

$$= \pm\sqrt{|\vec{p}|} \begin{pmatrix} \chi_\pm \\ \pm\chi_\pm \end{pmatrix} = \pm u_\pm(p). \tag{5.38}$$

A similar calculation shows that for the antiparticle solutions

$$\gamma_5 v_\pm(p) = \mp v_\pm(p). \tag{5.39}$$

Therefore, the helicity eigenstates for $m = 0$ are equivalent to the chirality eigenstates. Results (5.38) and (5.39) lead to the notion of handedness (which is borrowed from chemistry):

- u_+ describes a right handed particle: $\overrightarrow{\text{spin}} \upharpoonleft\upharpoonleft \vec{p}_{e^-}$ and
- v_+ describes a left handed antiparticle: $\overleftarrow{\text{spin}} \downharpoonleft\upharpoonleft \vec{p}_{e^+}$

where the converse holds for u_- and v_-.

Exploiting the eigenvalue equations (5.38) and (5.39), one can define the projectors

$$P_{\substack{R\\L}} = \frac{1}{2}(\mathbb{1} \pm \gamma_5). \tag{5.40}$$

They project to $u_\pm, v_\pm$ for arbitrary spinors. For example we have

$$P_L u_\pm = \frac{1}{2}(\mathbb{1} - \gamma_5)u_\pm = \frac{1}{2}(\mathbb{1} \mp \mathbb{1})u_\pm = \begin{cases} 0 \\ \mathbb{1}u_- \end{cases}.$$

To show that Eq. (5.40) indeed defines projectors, we check (using Eq. 5.36) for idempotence,

$$P^2_{\substack{R\\L}} = \frac{1}{4}(\mathbb{1} \pm \gamma_5)(\mathbb{1} \pm \gamma_5) = \frac{1}{4}(\mathbb{1} \pm 2\gamma_5 + \gamma_5^2) = \frac{1}{2}(\mathbb{1} \pm \gamma_5) = P_{\substack{R\\L}},$$

orthogonality,

$$P_R P_L = \frac{1}{4}(\mathbb{1} + \gamma_5)(\mathbb{1} - \gamma_5) = \frac{1}{4}(\mathbb{1} - \gamma_5^2) = 0,$$

and completeness,

$$P_R + P_L = \mathbb{1}.$$

Note that the projectors P_L and P_R are often used to indicate the chirality basis:

$$u_{L,R} = P_{L,R}u$$
$$v_{L,R} = P_{L,R}v.$$

What has been derived so far rests on the assumption that the mass be zero. In this case, chirality is equivalent to helicity which is also Lorentz invariant. If, on the other hand $m \neq 0$, chirality and helicity are not equivalent: In this case chirality, while Lorentz invariant, is not a constant of the motion,

$$[\gamma_5, H_{\text{Dirac}}] \neq 0,$$

and therefore not a good quantum number. Helicity though is a constant of the motion, but, since spin is unaffected by boosts, it is not Lorentz invariant for non-vanishing mass: For every possible momentum $\vec{p}$ in one frame of reference there is another frame in

Figure 5.1: *Helicity for the case of non-vanishing mass.*

		Chirality $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$	**Helicity** $h(\hat{\vec{p}}) = \frac{1}{2}\vec{\sigma}\cdot\hat{\vec{p}}\otimes\mathbb{1}$
$m = 0$	Constant of motion	✓	✓
	Lorentz invariant	✓	✓
$m \neq 0$	Constant of motion	✗	✓
	Lorentz invariant	✓	✗

Table 5.1: Chirality and helicity.

which the particle moves in direction $-\vec{p}/|\vec{p}|$ (see Fig. 5.1). A comparison of chirality and helicity is given in Tab. 5.1.

Although chirality is not a constant of the motion for $m \neq 0$, it is still a useful concept (and becomes important when one considers weak interactions). A solution of the Dirac equation ψ can be decomposed:

$$\psi = \psi_L + \psi_R$$

where ψ_L and ψ_R are not solutions of the Dirac equation. The W vector boson of the weak interaction only couples to ψ_L.

As for the normalization of the orthogonal spinors (5.29) and (5.30), the most convenient choice is:

$$\bar{u}_s(p)u_{s'}(p) = 2m\delta_{ss'}$$
$$\bar{v}_s(p)v_{s'}(p) = -2m\delta_{ss'}$$

where $s, s' = \pm$.

Using $\bar{\psi} = \psi^\dagger\gamma^0$, one can show that the following completeness relations (or polarization sum rules) hold:

$$\sum_{s=\pm} u_s(p)\bar{u}_s(p) = \not{p} + m \quad (5.41)$$

$$\sum_{s=\pm} v_s(p)\bar{v}_s(p) = \not{p} - m. \quad (5.42)$$

Comparing these polarization sums with, for instance, the Dirac equation for u, Eq. (5.27), one sees that $\not{p} + m$ projects on the subspace of particle solutions.

5.3 Field operator of the Dirac field

The spinors

$$u_s(p)e^{-ipx}, \text{ eigenvalues } E_p = +\sqrt{|\vec{p}|^2 + m^2}, \text{ and}$$
$$v_s(-p)e^{ipx}, \text{ eigenvalues } E_p = -\sqrt{|\vec{p}|^2 + m^2},$$

are eigenfunctions of the Dirac Hamiltonian and therefore solutions of the Dirac equation. From these solutions we can deduce the field operator of the Dirac field (which fulfills the Dirac equation) [1]:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p^0}} \sum_{s=\pm} \left\{ a_s(\vec{p}) u_s(p) e^{-ipx} + b_s^\dagger(\vec{p}) v_s(p) e^{ipx} \right\} \tag{5.43}$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p^0}} \sum_{s=\pm} \left\{ a_s^\dagger(\vec{p}) \bar{u}_s(p) e^{ipx} + b_s(\vec{p}) \bar{v}_s(p) e^{-ipx} \right\} \tag{5.44}$$

where

$a_s^\dagger(\vec{p})$: creation operator of particle with momentum $\vec{p}$
$b_s^\dagger(\vec{p})$: creation operator of antiparticle with momentum $\vec{p}$
$a_s(\vec{p})$: annihilation operator of particle with momentum $\vec{p}$
$b_s(\vec{p})$: annihilation operator of antiparticle with momentum $\vec{p}$.

In advanced quantum mechanics we have seen that field operators create or annihilate position eigenstates. The field operator in Eq. (5.44) does the same thing while furthermore consistently combining the equivalent possibilities for particle creation and antiparticle annihilation: $a_s^\dagger(\vec{p})$ creates individual particle momentum eigenstates from which a weighted superposition is formed, the integral over $b_s(\vec{p})$ on the other hand, annihilates a weighted superposition of antiparticles. Since the creation of a particle at position x is equivalent to the annihilation of its antiparticle at position x, both terms have to appear in the field operator $\bar{\psi}(x)$. Because we have to consider particles and antiparticles, here the energy spectrum is more complicated than in the pure particle case. The creation terms come with a positive-sign plane wave factor e^{ipx} while the annihilation terms contribute e^{-ipx}. The equivalence of particle creation and antiparticle annihilation is to be understood in the sense that they lead to the same change in a given field configuration.

The Dirac field is a spin-1/2 field. Therefore, the Pauli exclusion principle must hold, imposing anti-commutation relations on the field operators:

$$\{\psi(\vec{x},t), \psi(\vec{x}',t)\} = \{\bar{\psi}(\vec{x},t), \bar{\psi}(\vec{x}',t)\} = 0$$
$$\{\psi(\vec{x},t), \bar{\psi}(\vec{x}',t)\} = \gamma^0 \delta^3(\vec{x} - \vec{x}').$$

[1] The normalization is chosen to avoid an explicit factor $2p^0$ in the anticommutators of the fields and of the creation and annihilation operators.

Because of Eq. (5.43) and (5.44), this implies for the creation and annihilation operators

$$\begin{aligned}
\{a_r^\dagger(\vec{p}), a_s^\dagger(\vec{p}')\} &= \{a_r(\vec{p}), a_s(\vec{p}')\} = 0 \\
\{b_r^\dagger(\vec{p}), b_s^\dagger(\vec{p}')\} &= \{b_r(\vec{p}), b_s(\vec{p}')\} = 0 \\
\{a_r(\vec{p}), a_s^\dagger(\vec{p}')\} &= \delta_{rs}(2\pi)^3\delta^3(\vec{p}-\vec{p}') \\
\{b_r(\vec{p}), b_s^\dagger(\vec{p}')\} &= \delta_{rs}(2\pi)^3\delta^3(\vec{p}-\vec{p}').
\end{aligned}$$

As an example for the relation of field operator and ladder operator anti-commutation relations, we calculate $\{\psi(\vec{x},t), \bar{\psi}(\vec{x}',t)\}$, assuming anti-commutation relations for the creation and annihilation operators:

$$\begin{aligned}
&\{\psi(\vec{x},t), \bar{\psi}(\vec{y},t)\} \\
&= \int \frac{d^3p d^3\vec{q}}{(2\pi)^6} \frac{1}{\sqrt{2p^0 2q^0}} \sum_{r,s} \Big[e^{ipx} e^{-iqy} v_r(p)\bar{v}_s(q)\{b_r^\dagger(\vec{p}), b_s(\vec{q})\} \\
&\quad + e^{-ipx} e^{iqy} u_r(p)\bar{u}_s(q)\{a_r(\vec{p}), a_s^\dagger(\vec{q})\}\Big] \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \Big[e^{-i\vec{p}(\vec{x}-\vec{y})} \sum_s v_s(p)\bar{v}_s(p) + e^{i\vec{p}(\vec{x}-\vec{y})} \sum_s u_s(p)\bar{u}_s(p) \Big]
\end{aligned}$$

which, using the completeness relations, Eq. (5.41) and (5.42),

$$\begin{aligned}
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \Big[e^{-i\vec{p}(\vec{x}-\vec{y})} (\underbrace{p^0\gamma^0}_{\text{even}} - \underbrace{\vec{p}\cdot\vec{\gamma}}_{\text{odd}} - m) + e^{i\vec{p}(\vec{x}-\vec{y})} (\underbrace{p^0\gamma^0}_{\text{even}} - \underbrace{\vec{p}\cdot\vec{\gamma}}_{\text{odd}} + m) \Big] \\
&= \gamma^0 \int \frac{d^3p}{(2\pi^3)} e^{i\vec{p}(\vec{x}-\vec{y})} = \gamma^0 \delta^3(\vec{x}-\vec{y}).
\end{aligned}$$

However, in the laboratory one prepares in general (to a first approximation) momentum eigenstates, rather than position eigenstates. Therefore, we give the expression[2] for the momentum operator:

$$P^\mu = \int \frac{d^3k}{(2\pi)^3} k^\mu \sum_s \left(a_s^\dagger(\vec{k}) a_s(\vec{k}) + b_s^\dagger(\vec{k}) b_s(\vec{k}) \right)$$

which is just the momentum weighted with the number operator $N = a^\dagger a + b^\dagger b$. Using the anti-commutation relations for the ladder operators, one can show that the momentum operator fulfills the following useful commutation relations:

$$\begin{aligned}
[P^\mu, a_s^\dagger(\vec{p})] &= p^\mu a_s^\dagger(\vec{p}) \\
[P^\mu, b_s^\dagger(\vec{p})] &= p^\mu b_s^\dagger(\vec{p}) \\
[P^\mu, a_s(\vec{p})] &= -p^\mu a_s(\vec{p}) \\
[P^\mu, b_s(\vec{p})] &= -p^\mu b_s(\vec{p}).
\end{aligned}$$

[2] This expression is obtained from Noether's theorem using the technique of normal ordering. These topics are discussed in text books on quantum field theory, e. g. by Peskin/Schroeder [14].

Vacuum state The vacuum state is denoted by $|0\rangle$ and has the property [3],

$$P^\mu |0\rangle = 0, \tag{5.45}$$

i.e. the vacuum has no momentum.

Using the commutation relations stated above and the property (5.45), we conclude that,

$$P^\mu a_s^\dagger(\vec{p})\, |0\rangle = p^\mu a_s^\dagger(\vec{p})\, |0\rangle\,, \tag{5.46}$$

in other words, the state $a_s^\dagger(\vec{p})\,|0\rangle$ is an eigenstate of P^μ with momentum p^μ.

With this fact in mind, we define the following states,

$$\left|e^-(p,s)\right\rangle = \sqrt{2E_{\vec{p}}}\, a_s^\dagger(\vec{p})\, |0\rangle\,, \tag{5.47}$$

$$\left|e^+(p,s)\right\rangle = \sqrt{2E_{\vec{p}}}\, b_s^\dagger(\vec{p})\, |0\rangle\,, \tag{5.48}$$

of a particle respectively antiparticle with momentum eigenstate p and spin s.

The factor $\sqrt{2E_{\vec{p}}}$ is there in order to ensure a Lorentz invariant normalization,

$$\begin{aligned}
\langle e^-(q,r)|e^-(p,s)\rangle &= 2\sqrt{E_{\vec{q}}E_{\vec{p}}}\,\langle 0|\, a_r(\vec{q})a_s^\dagger(\vec{p})\,|0\rangle \\
&= 2\sqrt{E_{\vec{q}}E_{\vec{p}}}\,\langle 0|\left\{a_r(\vec{q}), a_s^\dagger(\vec{p})\right\} - a_s^\dagger(\vec{p})\,\underbrace{a_r(\vec{q})\,|0\rangle}_{=0} \\
&= \delta_{rs} 2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{q}-\vec{p}).
\end{aligned}$$

The definition of states (5.47) and (5.48) corresponds to a continuum normalization in infinite volume. From the above equation, it can be seen that the dimensionality of the one-particle norm $\langle e^-(p,s)|e^-(p,s)\rangle$ is,

$$\frac{(energy)}{(momentum)^3} = (energy)\cdot(volume),$$

meaning that we have a constant particle density of $2E$ particles per unit volume. To obtain single particle states in a given volume V, one must therefore multiply $|e^-(p,s)\rangle$ with a normalization factor $1/\sqrt{2EV}$:

$$\left|e^-(p,s)\right\rangle_{\text{single-particle}} = \frac{1}{\sqrt{2EV}}\left|e^-(p,s)\right\rangle \tag{5.49}$$

$$\left|e^+(p,s)\right\rangle_{\text{single-particle}} = \frac{1}{\sqrt{2EV}}\left|e^+(p,s)\right\rangle \tag{5.50}$$

[3] After applying the nontrivial concept of normal ordering, here only motivated by the number interpretation in the operator P^μ.

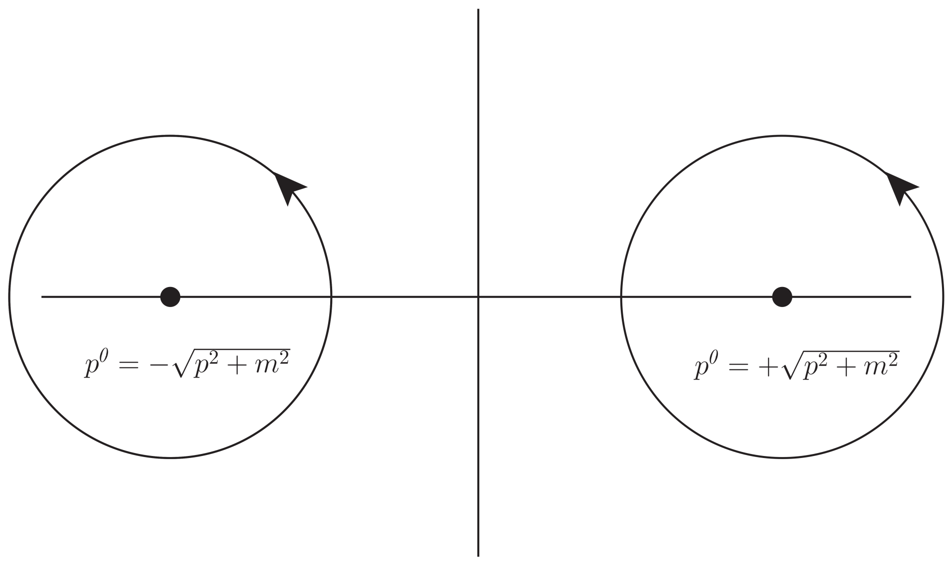


Figure 5.2: *Integration paths for Dirac propagator.*

5.4 Dirac propagator

In order to solve general Dirac equations, we want to apply a formalism similar to the one used in classical electrodynamics, namely Green's functions.

We introduce the scalar propagator,

$$\begin{aligned}\Delta^{\pm}(x) &= \pm\frac{1}{i}\int\frac{d^3p}{(2\pi)^3 2p^0}\mathrm{e}^{\mp ip\cdot x}\\ &= \pm\frac{1}{i}\int\frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\mathrm{e}^{\mp ip\cdot x},\end{aligned} \tag{5.51}$$

which satisfies the Klein-Gordon equation,

$$(\Box + m^2)\Delta^{\pm}(x) = 0.$$

Representation as a contour integral

$$\Delta^{\pm}(x) = -\int_{C^{\pm}}\frac{d^4p}{(2\pi)^4}\frac{\mathrm{e}^{-ip\cdot x}}{p^2-m^2}, \tag{5.52}$$

where the paths $C^{\pm}$ are depicted in Fig. 5.2.

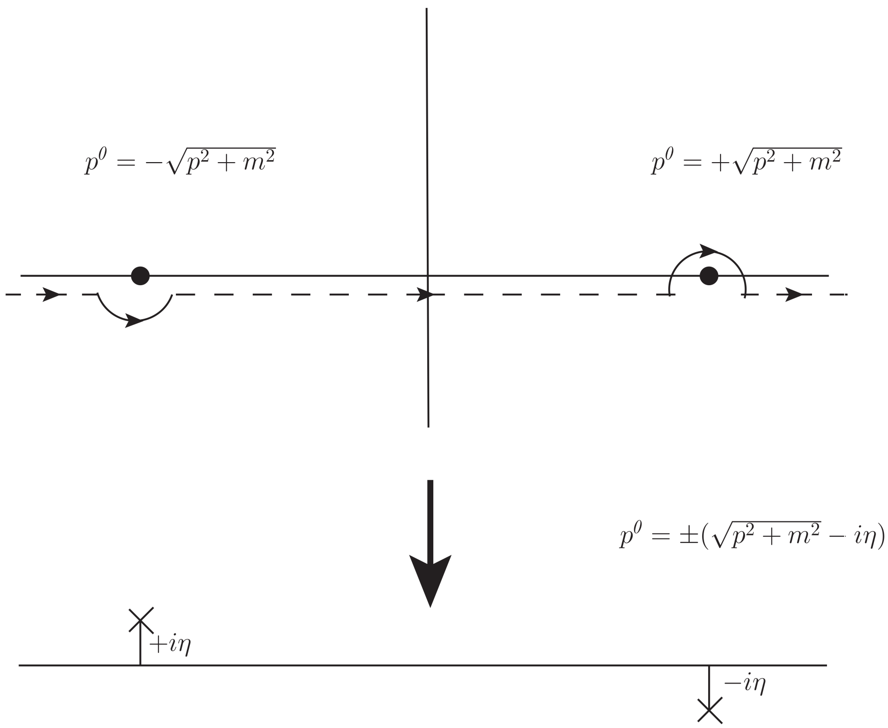


Figure 5.3: *Deformed integration paths and $+i\varepsilon$ convention.*

5.4.1 Feynman propagator

To get a "true" Green's function for the operator $\Box + m^2$, we need to introduce a discontinuity, and define the Feynman propagator

$$\Delta_F(x) = \theta(t)\Delta^+(x) - \theta(-t)\Delta^-(x), \tag{5.53}$$

where we deform the paths of Fig. 5.3 according to the sign of $t = x^0$ to get convergent integrals over the real line (details can be found in a complex analysis book, see e.g. Freitag & Busam [15]) :

- $x^0 > 0, \operatorname{Im} p^0 < 0 \Rightarrow \mathrm{e}^{-ip^0x^0} \overset{R\to\infty}{\longrightarrow} 0 : C^+$,
- $x^0 < 0, \operatorname{Im} p^0 > 0 \Rightarrow \mathrm{e}^{-ip^0x^0} \overset{R\to\infty}{\longrightarrow} 0 : C^-$.

$+i\varepsilon$ convention Instead of deforming the integration path, one can also shift the two poles and integrate over the whole real p^0-axis, without having to worry about the poles,

$$p^0 = \pm\sqrt{\vec{p}^{\,2} + m^2} \longrightarrow \pm(\sqrt{\vec{p}^{\,2} + m^2} - i\eta),$$

yielding

$$\Delta_F(x) = \lim_{\varepsilon\to 0+} \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\varepsilon}, \tag{5.54}$$

the Green's function of the Klein-Gordon equation,

$$(\Box + m^2)\Delta_F(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{-p^2 + m^2}{p^2 - m^2} = -\delta^{(4)}(x). \tag{5.55}$$

Propagator A propagator is the transition amplitude of a particle between creation at x^μ and annihilation at x'^μ (or vice-versa). It is a fundamental tool of quantum field theory.

After getting the Feynman propagator for the Klein-Gordon field (spin 0), we want to focus on the propagator for fermions (spin 1/2).

We compute the anticommutation relations for the field in this case getting,

$$\begin{aligned}\{\psi(x), \bar{\psi}(x')\} =& \int \frac{d^3p d^3p'}{(2\pi)^6\sqrt{2p^0}\sqrt{2p'^0}} \sum_{r,s} \Big[e^{i(p\cdot x - p'\cdot x')} v_r(p)\bar{v}_s(p')\{b_r^\dagger(p), b_s(p')\} \\ & e^{-i(p\cdot x - p'\cdot x')} u_r(p)\bar{u}_s(p')\{a_r(p), a_s^\dagger(p')\}\Big] \\ =& \int \frac{d^3p}{(2\pi)^3 2p^0} \left[e^{ip\cdot(x-x')}(\not{p} - m) + e^{-ip\cdot(x-x')}(\not{p} + m)\right] \\ =& (i\not{\partial} + m) \int \frac{d^3p}{(2\pi)^3 2p^0} \left(e^{-ip\cdot(x-x')} - e^{ip\cdot(x-x')}\right),\end{aligned} \tag{5.56}$$

where we made use of the completeness relations (5.41) and (5.42) in going from the first to the second line.

We now define the **Feynman fermion propagator**,

$$iS(x - x') \equiv (i\not{\partial} + m)(\Delta^+(x - x') + \Delta^-(x - x')). \tag{5.57}$$

Splitting ψ and $\bar{\psi}$ in their creation $\psi^-, \bar{\psi}^-$ and annihilation $\psi^+, \bar{\psi}^+$ parts (looking only at the operators $a_s^\dagger, b_s^\dagger$ and a_s, b_s respectively), we get the comutation relations,

$$\{\psi^+(x), \bar{\psi}^-(x')\} = (i\not{\partial} + m)\Delta^+(x - x') = iS^+(x - x'), \tag{5.58}$$
$$\{\psi^-(x), \bar{\psi}^+(x')\} = (i\not{\partial} + m)\Delta^-(x - x') = iS^-(x - x'). \tag{5.59}$$

$S^\pm(x - x')$ can as well be represented as contour integrals,

$$S^\pm(x) = \int_{C^\pm} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{\not{p} + m}{p^2 - m^2} = \int_{C^\pm} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{1}{\not{p} - m}, \tag{5.60}$$

which is well defined because $(\not p + m)(\not p - m) = (p^2 - m^2)\mathbb{1}$.

We take a look at the time ordered product of fermion operators,

$$T(\psi(x)\bar\psi(x')) = \begin{cases} \psi(x)\bar\psi(x'), & t > t' \\ -\bar\psi(x')\psi(x), & t' > t \end{cases}$$
$$= \theta(t - t')\psi(x)\bar\psi(x') - \theta(t' - t)\bar\psi(x')\psi(x).$$

The Feynman fermion propagator is then the vacuum expectation value of this time ordered product,

$$iS_F(x - x') = \langle 0| T(\psi(x)\bar\psi(x')) |0\rangle . \tag{5.61}$$

Remembering the destroying effect of annihilation operators on the vacuum, we can skip some trivial steps of the calculation. We look separately at both time ordering cases, getting,

$$\langle 0| \psi(x)\bar\psi(x') |0\rangle = \langle 0| \psi^+(x)\bar\psi^-(x') |0\rangle = \langle 0| \{\psi^+(x), \bar\psi^-(x')\} |0\rangle = iS^+(x - x'),$$
$$\langle 0| \bar\psi(x')\psi(x) |0\rangle = \langle 0| \bar\psi^+(x')\psi^-(x) |0\rangle = \langle 0| \{\bar\psi^+(x'), \psi^-(x)\} |0\rangle = iS^-(x - x'),$$

yielding,

$$S_F(x) = \theta(t)S^+(x) - \theta(-t)S^-(x) = (i\not\partial + m)\Delta_F(x), \tag{5.62}$$

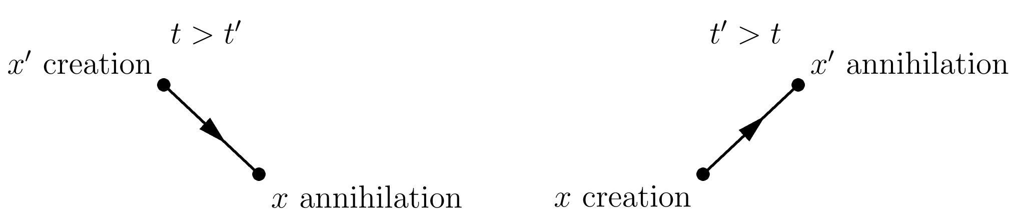
or, as a contour integral,

$$S_F(x) = \int_{C_F} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{1}{\not p - m} = \lim_{\varepsilon\to 0+} \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{\not p + m}{p^2 - m^2 + i\varepsilon}. \tag{5.63}$$

We then see that the fermion propagator is nothing else than the Green's function of the Dirac equation,

$$(i\not\partial - m)S_F(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{(\not p - m)(\not p + m)}{p^2 - m^2} = \delta^{(4)}(x)\mathbb{1}. \tag{5.64}$$

The interpretation of S_F is then similar to the one of the Green's function in classical electrodynamics:



We can ask ourselves why the time ordering procedure is important. In scattering processes both orderings are not distinguishable (see Fig. 5.4) in experiments, so that we can understand as a sum over both time ordering possibilities.

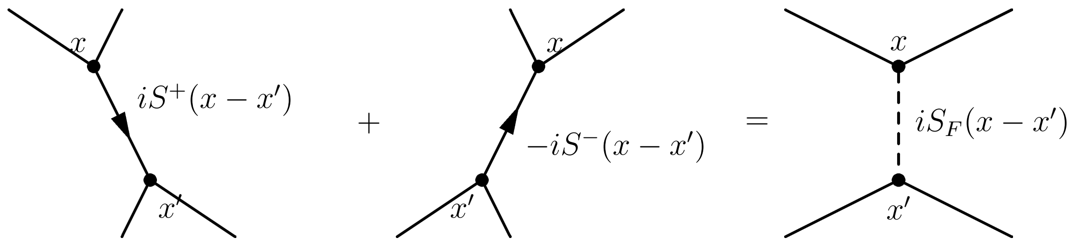


Figure 5.4: Sum of both time orderings

5.5 Photon field operator

After being able to describe free scalar fields (Klein-Gordon, spin 0) and free fermion fields (Dirac, spin 1/2), we go on to vector fields (spin 1) like the one describing the photon. The photon field will be shown to have a fundamental importance in QED since it is the interaction field between fermions.

To start, we recall the photon field operator of advanced quantum mechanics, which reads in Coulomb gauge,

$$\vec{A}(x) = \sum_{\alpha=1,2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(a_\alpha(\vec{k})\,\vec{\varepsilon}_\alpha(\vec{k})\mathrm{e}^{-ik\cdot x} + a^\dagger_\alpha(\vec{k})\,\vec{\varepsilon}^{\,*}_\alpha(\vec{k})\mathrm{e}^{ik\cdot x} \right). \tag{5.65}$$

In Eq. (5.65), $a^\dagger_\alpha(\vec{k})$ creates a photon of momentum $\vec{k}$ and polarization α, and $a_\alpha(\vec{k})$ destroys the same.

Since we are dealing with a bosonic field, we impose the commutation relations,

$$[a_\alpha(\vec{k}), a^\dagger_\beta(\vec{k}')] = -g_{\alpha\beta}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}'), \tag{5.66}$$

$$[a_\alpha(\vec{k}), a_\beta(\vec{k}')] = [a^\dagger_\alpha(\vec{k}), a^\dagger_\beta(\vec{k}')] = 0. \tag{5.67}$$

Supposing that the photon propagates in the z-direction ($k^\mu = (k,0,0,k)^\top$), we have the following possibilites for the polarization vectors :

- linear : $\varepsilon_1^\mu = (0,1,0,0)^\top, \varepsilon_2^\mu = (0,0,1,0)^\top$,
- circular : $\varepsilon_+^\mu = \frac{1}{\sqrt{2}}(\varepsilon_1^\mu + i\varepsilon_2^\mu) = \frac{1}{\sqrt{2}}(0,1,i,0)^\top, \varepsilon_-^\mu = \frac{1}{\sqrt{2}}(\varepsilon_1^\mu - i\varepsilon_2^\mu) = \frac{1}{\sqrt{2}}(0,1,-i,0)^\top$.

These vector sets satisfy the completness relation,

$$\Pi^{\mu\nu} = \sum_{\substack{\lambda=\pm \\ (\text{or } \lambda=1,2)}} \varepsilon^{*\mu}_\lambda \varepsilon^\nu_\lambda = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}. \tag{5.68}$$

By applying a well chosen boost to $\Pi^{\mu\nu}$ we can easily check that it is in general not Lorentz invariant. We have to choose a specific gauge depending on the reference frame, parametrized by a real number n.

To do so we define a auxiliary vector $n^\mu = n(1,0,0,-1)^\top$ satisfying $n_\sigma k^\sigma = 2kn$ and get the "axial gauge",

$$\Pi^{\mu\nu} = -g^{\mu\nu} + \frac{n^\mu k^\nu + k^\mu n^\nu}{n_\sigma k^\sigma}. \tag{5.69}$$

For $n = 1$, we recover the Coulomb gauge,

$$\Pi^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

In physical processes, the photon field couples to an external current,

$$j^\mu(x) = j^\mu(k)\mathrm{e}^{ik\cdot x},$$

and we have the current conservation,

$$\partial_\mu j^\mu = 0,$$

which yields in Fourier space,

$$k_\mu j^\mu = 0,$$

and thus,

$$j_\mu \Pi^{\mu\nu} = j_\nu \Pi^{\mu\nu} = 0,$$

i.e. the $n^\mu k^\nu + k^\mu n^\nu$ term vanishes when contracted with external currents, such that we are left with an effective polarization sum,

$$p^{\mu\nu}_{\mathit{eff}} = -g^{\mu\nu}. \tag{5.70}$$

We now look at the time ordered product of photon field operators,

$$T(A_\mu(x)A_\nu(x')) = \begin{cases} A_\mu(x)A_\nu(x'), & t > t' \\ A_\nu(x')A_\mu(x), & t' > t \end{cases}. \tag{5.71}$$

Repeating the same steps as in the fermion case, we get the **photon propagator**,

$$iD_{F,\mu\nu}(x - x') = \langle 0|\, T(A_\mu(x)A_\nu(x'))\, |0\rangle \tag{5.72}$$

$$= -ig_{\mu\nu}\Delta_F(x - x') \tag{5.73}$$

$$= -ig_{\mu\nu} \lim_{\varepsilon\to 0+} \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{e}^{-ik\cdot x}}{k^2 + i\varepsilon}. \tag{5.74}$$

Finally, we see that the photon propagator is the Green's function of the wave equation,

$$\Box D_{F,\mu\nu}(x) = g_{\mu\nu}\delta^{(4)}(x). \tag{5.75}$$

5.6 Interaction representation

In the previous sections, we have gained an understanding of the free fields occurring in QED. The next step is to introduce a way to handle interactions between those fields.

Idea: decompose the Hamiltonian in the Schrödinger representation,

$$H_S = H_{0,S} + H'_S,$$

and define states and operators in the free Heisenberg representation,

$$\begin{aligned} \psi_I &= \mathrm{e}^{iH_{0,S}t}\psi_S \\ O_I &= \mathrm{e}^{iH_{0,S}t}O_S\mathrm{e}^{-iH_{0,S}t}, \end{aligned}$$

and you get the **interaction representation** (also called **Dirac representation**).

We have, in particular,

$$H_{0,I} = H_{0,S} = H_0, \tag{5.76}$$

and the time evolution of ψ_I respectively O_I becomes,

$$i\partial_t\psi_I = H'_I\psi_I, \tag{5.77}$$

$$i\partial_t O_I = -H_0 O_I + O_I H_0 = [O_I, H_0], \tag{5.78}$$

i.e. ψ_I is influenced only by the "true" interaction part; the "trivial" time evolution (free part) has been absorbed in the operators O_I.

Comparison The Schrödinger, Heisenberg, and interaction representations differ in the way they describe time evolution:

- Schrödinger representation: states contain time evolution, operators are time independent;
- Heisenberg representation: states are time independent, operators contain time evolution;
- Interaction representation: time dependence of states only due to interactions, free (also called "trivial") time evolution for operators.

This comparison shows that the interaction representation is a mixture of both other representations.

5.6.1 Time evolution operator

In preparation for time-dependent perturbation theory, we consider the time evolution operator $U(t, t_0)$ in the interaction representation:

$$\psi_I(t) = U(t, t_0)\psi_I(t_0). \tag{5.79}$$

The time evolution operator in Eq. (5.79) can be written in terms of the free and interaction Hamiltonians, Eq. (5.76), in the Schrödinger representation by using the time evolution properties:

$$\psi_I(t) = e^{iH_0t}\psi_S(t) = e^{iH_0t}e^{-iH_S(t-t_0)}\psi_S(t_0) = e^{iH_0t}e^{-iH_S(t-t_0)}e^{-iH_0t_0}\psi_I(t_0).$$

Comparing this result with Eq. (5.79) yields

$$U(t, t_0) = e^{iH_0t}e^{-iH_S(t-t_0)}e^{-iH_0t_0}. \tag{5.80}$$

An interaction picture operator is related by

$$O_H(t) = U^\dagger(t, t_0)O_I U(t, t_0)$$

to its Heisenberg picture equivalent.

Because of Eq. (5.80) the time evolution operator has the following properties:

- $U(t_0, t_0) = \mathbb{1}$,
- $U(t_2, t_1)U(t_1, t_0) = U(t_2, t_0)$,
- $U^{-1}(t_0, t_1) = U(t_1, t_0)$, and
- $U^\dagger(t_1, t_0) = U^{-1}(t_1, t_0) = U(t_0, t_1)$.

5.6.2 Time ordering

To find the time evolution operator, the time evolution (Schrödinger) equation

$$i\frac{\partial}{\partial t}U(t, t_0) = H_I' U(t, t_0) \tag{5.81}$$

has to be solved. This is equivalent to the integral equation

$$U(t, t_0) = \mathbb{1} + (-i)\int_{t_0}^{t} dt_1 H_I'(t_1)U(t_1, t_0)$$

which can be iterated to give the Neumann series

$$U(t,t_0) = \mathbb{1} + (-i)\int_{t_0}^{t} dt_1 H_I'(t_1) \tag{5.82}$$

$$+(-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I'(t_1) H_I'(t_2) \tag{5.83}$$

$$+\dots \tag{5.84}$$

$$+(-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n H_I'(t_1) \dots H_I'(t_n). \tag{5.85}$$

This is not yet satisfactory since the boundary of every integral but the first depends on the foregoing integration. To solve this problem, one uses time ordering. Let us first consider the following identities:

$$\begin{aligned}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I'(t_1) H_I'(t_2) &= \int_{t_0}^{t} dt_2 \int_{t_2}^{t} dt_1 H_I'(t_1) H_I'(t_2) \\ &= \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 H_I'(t_2) H_I'(t_1)\end{aligned}$$

where in the first line the integration domains are identical (see Fig. 5.5) and in going to the second line the variable labels are exchanged. We can combine these terms in a more compact expression:

$$\begin{aligned}&2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I'(t_1) H_I'(t_2) \\ &= \int_{t_0}^{t} dt_2 \int_{t_2}^{t} dt_1 H_I'(t_1) H_I'(t_2) + \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 H_I'(t_2) H_I'(t_1) \\ &= \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \Big(H_I'(t_1) H_I'(t_2) \theta(t_1 - t_2) + H_I'(t_2) H_I'(t_1) \theta(t_2 - t_1) \Big) \\ &= \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T\Big(H_I'(t_1) H_I'(t_2) \Big).\end{aligned}$$

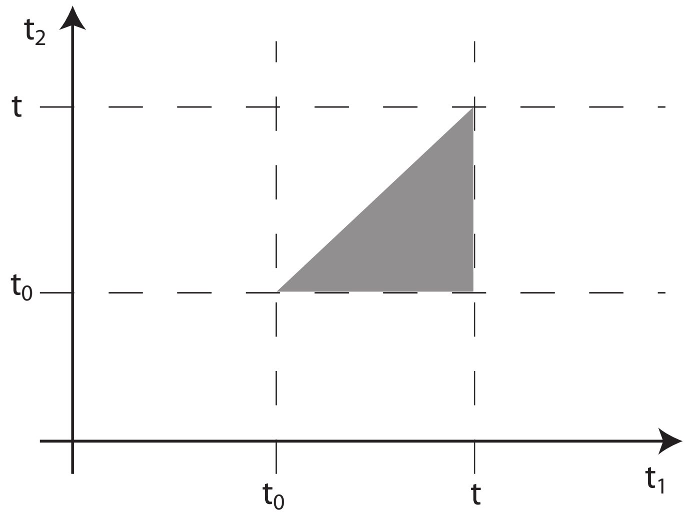


Figure 5.5: *Identical integration domains.*

All terms of the Neumann series can be rewritten in this way. For the n-th term in Eq. (5.85) we have

$$n!\int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t_{n-1}} dt_n H_I'(t_1)\ldots H_I'(t_n)$$
$$= \int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_n T\Big(H_I'(t_1)\ldots H_I'(t_n)\Big).$$

We therefore obtain the following perturbation series[4] for the time evolution operator:

$$U(t,t_0) = \sum_{n=0}^{\infty} \frac{1}{n!}(-i)^n \int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_n T\Big(H'(t_1)\ldots H'(t_n)\Big). \tag{5.86}$$

Defining the time ordered exponential, Eq. (5.86) can be written as

$$\boxed{U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} dt' H'(t')\right)}. \tag{5.87}$$

[4]We are working in the interaction picture and drop the index I for simplicity.

We check that this result indeed solves the time evolution equation (5.81):

$$\begin{aligned}
i\frac{\partial}{\partial t}U(t,t_0) &= i\sum_{n=1}^{\infty}\frac{1}{n!}(-i)^n n\int_{t_0}^{t}dt_1\ldots\int_{t_0}^{t}dt_{n-1} \\
&\qquad T\Big(H'(t_0)\ldots H'(t_{n-1})H'(t)\Big) \\
&= H'(t)\sum_{n=1}^{\infty}\frac{1}{(n-1)!}(-i)^{n-1}\int_{t_0}^{t}dt_1\ldots\int_{t_0}^{t}dt_{n-1} \\
&\qquad T\Big(H'(t_0)\ldots H'(t_{n-1})\Big) \\
&= H'(t)U(t,t_0).
\end{aligned}$$

5.7 Scattering matrix

Our overall aim is to develop a formalism to compute scattering matrix elements which describe the transition from initial states defined at $t \to -\infty$ to final states observed at $t \to +\infty$. To this end, we split up the Hamiltonian into a solvable free part which determines the operators' time evolution and an interaction part responsible for the time evolution of the states. Now we investigate how the time ordered exponential that is the time evolution operator, see Eq. (5.87), relates to the $\mathcal{S}$-matrix.

The scattering matrix element $\langle f|\,\mathcal{S}\,|i\rangle$ is the transition amplitude for $|i\rangle \to |f\rangle$ caused by interactions. The state of the system is described by the time dependent state vector $|\psi(t)\rangle$. The above statement about asymptotically large times can now be recast in a more explicit form: The initial state is given by

$$\lim_{t\to-\infty}|\psi(t)\rangle = |\phi_i\rangle$$

where $|\phi_i\rangle$ is an eigenstate of the free Hamilton operator and $t \to -\infty$ is justified since the interaction timescale is about 10^{-15} s. The scattering matrix element $\mathcal{S}_{fi}$ is given by the projection of the state vector $|\psi(t)\rangle$ onto a final state $|\phi_f\rangle$:

$$\mathcal{S}_{fi} = \lim_{t\to+\infty}\langle\phi_f|\psi(t)\rangle = \langle\phi_f|\,\mathcal{S}\,|\phi_i\rangle\,.$$

Using the time evolution operator (and its action on a state, see Eq. (5.79)), this can be expressed as

$$\mathcal{S}_{fi} = \lim_{t_2\to+\infty}\lim_{t_1\to-\infty}\langle\phi_f|\,U(t_2,t_1)\,|\phi_i\rangle\,.$$

We can therefore conclude that

$$\boxed{\mathcal{S} = U(+\infty,-\infty)} = \sum_{n=0}^{\infty}\frac{1}{n!}(-i)^n\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_n T\Big(H'(t_1)\ldots H'(t_n)\Big). \tag{5.88}$$

As an instructive example, we consider $2 \to 2$ scattering:

$$k_1 + k_2 \to k_3 + k_4.$$

The scattering matrix element is given by

$$\mathcal{S}_{fi} = \langle f| \mathcal{S} |i\rangle = \underbrace{\langle 0| a(k_4)a(k_3)}_{\langle \phi_f|} |\mathcal{S}| \underbrace{a^\dagger(k_1)a^\dagger(k_2) |0\rangle}_{|\phi_i\rangle}.$$

The $\mathcal{S}$-operator itself consists of further creation and annihilation operators belonging to further quantum fields. By evaluation of the creators and annihilators in $\mathcal{S}$ (using commutation or anticommutation relations), it follows that there is only one single non-vanishing contribution to $\mathcal{S}_{fi}$ being of the ("normally ordered") form

$$f(k_1, k_2, k_3, k_4)a^\dagger(k_3)a^\dagger(k_4)a(k_2)a(k_1).$$

Note that in the above expression, the annihilation operators stand on the right hand side, while the creation operators are on the left. Such expressions are said to be in normal order and are denoted by colons, $: ABC :$. Since the aim is to find the non-vanishing contributions, a way has to be found how time ordered products can be related to products in normal order. For instance, consider the time ordered product of two Boson field operators (where $A^+,\ B^+$ are annihilators and $A^-,\ B^-$ creators)[5]

$$\begin{aligned} T\Big(A(x_1)B(x_2)\Big)\Big|_{t_1>t_2} &= A(x_1)B(x_2) \\ &= A^+(x_1)B^+(x_2) + A^-(x_1)B^+(x_2) \\ &+ \underbrace{A^+(x_1)B^-(x_2)}_{\text{not in normal order}} + A^-(x_1)B^-(x_2). \end{aligned}$$

[5] The $\pm$ sign is motivated by the decomposition of field operators in positive and negative frequency parts:

$$\phi(x) = \phi^+(x) + \phi^-(x).$$

Consider for example the Klein-Gordon field where

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p^0}} \Big(a(p)e^{+i\vec{p}\cdot\vec{x}} + a^\dagger(p)e^{-i\vec{p}\cdot\vec{x}} \Big)$$

and therefore

$$\begin{aligned} \phi^+(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p^0}} a(p)e^{+i\vec{p}\cdot\vec{x}} \\ \phi^-(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p^0}} a^\dagger(p)e^{-i\vec{p}\cdot\vec{x}}. \end{aligned}$$

One can observe that only one of the above terms is not in normal order while the other three would vanish upon evaluation in $\langle 0| \cdot |0\rangle$. Using

$$\underbrace{A^+(x_1)B^-(x_2)}_{\text{not in normal order}} = \underbrace{B^-(x_2)A^+(x_1)}_{\text{in normal order}} + \underbrace{[A^+(x_1), B^-(x_2)]}_{\text{c-number}},$$

we rewrite

$$\begin{aligned}
[A^+(x_1), B^-(x_2)] &= \langle 0| \, [A^+(x_1), B^-(x_2)] \, |0\rangle \\
&= \langle 0| \, A^+(x_1)B^-(x_2) \, |0\rangle \\
&= \langle 0| \, T(A(x_1)B(x_2)) \, |0\rangle \, .
\end{aligned}$$

Since the same holds for $t_1 < t_2$, we draw the conclusion

$$T\Big(A(x_1)B(x_2)\Big) = :A(x_1)B(x_2): + \langle 0| \, T(A(x_1)B(x_2)) \, |0\rangle \, .$$

An analogous calculation for fermion operators yields the same result.

The next step towards Feynman diagrams is to formalize this connection between time and normal ordered products. We first define the following shorthand

$$\underbracket{\phi_A(x_1)\phi_B}(x_2) = \langle 0| \, T(\phi_A(x_1)\phi_B(x_2)) \, |0\rangle$$

which is called contraction of operators. This allows to state the following in compact notation.

Wick's theorem: The time ordered product of a set of operators can be decomposed into the sum of all corresponding contracted products in normal order. All combinatorially allowed contributions appear:

$$\begin{aligned}
&T(ABC \ldots XYZ) = :ABC \ldots XYZ: \\
&+ :\underbracket{AB}\,C \ldots XYZ: + \cdots + :\underbracket{ABC \ldots XYZ}: + \cdots + :ABC \ldots X\,\underbracket{YZ}: \\
&+ :\underbracket{AB}\,\underbracket{CD} \ldots XYZ: + :\underbracket{A\underbracket{BC}D} \ldots XYZ: + \ldots \\
&+ :\text{threefold contractions}: + \ldots \; .
\end{aligned}$$

5.8 Feynman rules of quantum electrodynamics

The Lagrangian density of QED is given by

$$\mathcal{L} = \mathcal{L}_0^{\text{Dirac}} + \mathcal{L}_0^{\text{photon}} + \mathcal{L}'$$

where the subscript 0 denotes the free Lagrangian densities and $'$ denotes the interaction part. In particular, we have

$$\mathcal{L}_0^{\text{Dirac}} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{5.89}$$

$$\mathcal{L}_0^{\text{photon}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{5.90}$$

$$\mathcal{L}' = -e\bar{\psi}\gamma_\mu\psi A^\mu = -j_\mu A^\mu \tag{5.91}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Note that from $\mathcal{L}_0^{\text{photon}}$ the free Maxwell's equations can be derived using the Euler-Lagrange equations. Using $\mathcal{L}_0^{\text{photon}} + \mathcal{L}'$ yields Maxwell's equations in the presence of sources and $\mathcal{L}_0^{\text{Dirac}} + \mathcal{L}'$ does the same for the Dirac equation. The interaction term $\mathcal{L}'$ describes current-field interactions and therefore couples the fermions described by the Dirac equation to photons described by Maxwell's equations.

Using Eq. (5.91), one finds the quantized interaction Hamiltonian density

$$\mathcal{H}' = -\mathcal{L}' = e\bar{\psi}\gamma_\mu\psi A^\mu.$$

Integrating the interaction Hamiltonian density over all space yields the interaction Hamiltonian,

$$H' = \int d^3\vec{x}\,\mathcal{H}',$$

and, in the integral representation of $\mathcal{S}$ given in Eq. (5.88), this leads to integrations over space-time:

$$\mathcal{S} = \sum_{n=0}^{\infty}\frac{1}{n!}(-ie)^n\int d^4x_1\ldots d^4x_n T\Big(\bar{\psi}(x_1)\gamma_{\mu_1}\psi(x_1)A^{\mu_1}\ldots\bar{\psi}(x_n)\gamma_{\mu_n}\psi(x_n)A^{\mu_n}\Big). \tag{5.92}$$

Since $e = \sqrt{4\pi\alpha}$ (see Eq. (1.9)), the coupling constant appears in the interaction term and n-th order terms are suppressed with e^n. This means that we found an expansion of $\mathcal{S}$ in the small parameter e which is the starting point for perturbation theory. The structure of the n-th term in the perturbation series in Eq. (5.92) is

$$\mathcal{S}^{(n)} = \frac{1}{n!}\int d^4x_1\ldots d^4x_n\mathcal{S}_n \tag{5.93}$$

where

$$\mathcal{S}_n = \sum_{\text{contractions}} K(x_1,\ldots,x_n)\ :\ldots\bar{\psi}(x_i)\ldots\psi(x_j)\ldots A(x_n):. \tag{5.94}$$

For a specific scattering process, the relevant matrix element is

$$\mathcal{S}_{fi} = \underbrace{\langle f|}_{\sim a}\,\mathcal{S}\,\underbrace{|i\rangle}_{\sim a^\dagger}$$

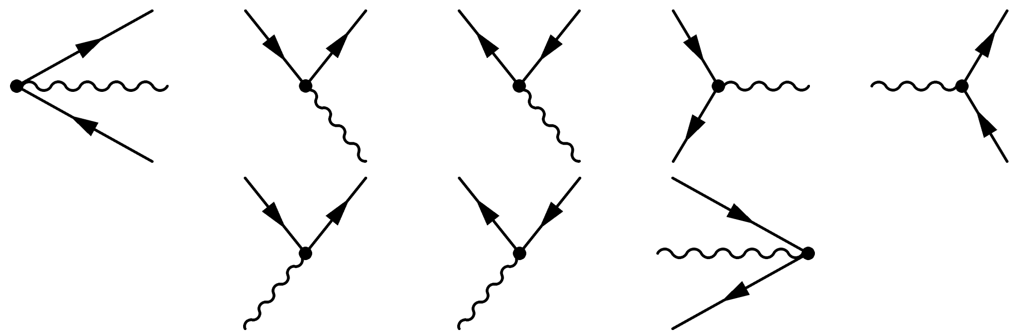
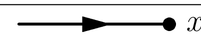
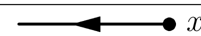
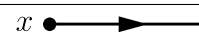
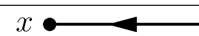
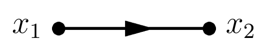
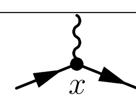


Figure 5.6: *First order contributions* $\mathcal{S}^{(1)}$. These processes violate energy-momentum conservation and are therefore unphysical.

which means that only terms in $\mathcal{S}$ matching $\langle f| \cdot |i\rangle$ yield contributions to the transition amplitude. The following field operators, which constitute the Feynman rules in position space, are contained in $\mathcal{S}$ ($\overrightarrow{\text{time}}$).

$\psi^+(x)$	absorption of electron at x	
$\bar{\psi}^+(x)$	absorption of positron at x	
$\bar{\psi}^-(x)$	emission of electron at x	
$\psi^-(x)$	emission of positron at x	
$A^+(x)$	absorption of photon at x	
$A^-(x)$	emission of photon at x	
$\underbrace{\psi(x_2)\bar{\psi}(x_1)}$ $= iS_F(x_2 - x_1)$	Fermion propagator	
$\underbrace{A^\mu(x_2)A^\nu}(x_1)$ $= iD_F^{\mu\nu}(x_2 - x_1)$	photon propagator	
$-ie\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x)$ $= -ie\gamma_\mu \cdot$ vertex at x	vertex at x	

The $\mathcal{S}$-operator at order n is examined using Wick's theorem. At fist order, this yields (remembering Eq. (5.92) while ignoring disconnected contributions from Wick's theorem) the following $2^3 = 8$ contributions:

$$\mathcal{S}^{(1)} = -ie \int d^4x T(\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x)) = -ie \int d^4x \,{:}\,\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x)\,{:}\,.$$

There is a total of 8 possible combinations, since A^μ creates or annihilates a photon, $\bar{\psi}$ creates an electron or annihilates a positron, and ψ creates a position or annihilates an electron. Fig. 5.6 shows the corresponding Feynman diagrams.

However, all these processes are unphysical because they violate energy-momentum conservation:

$$\pm p_{e^+} \pm p_{e^-} \pm p_\gamma \neq 0$$

which is because free particles fulfill

$$p_{e^+}^2 = m_e^2 \qquad p_{e^-}^2 = m_e^2 \qquad p_\gamma^2 = 0.$$

To find physical contributions to the interaction Hamiltonian, we turn to the second order contributions to $\mathcal{S}$ (see Eq. (5.92)):

$$\mathcal{S}^{(2)} = \frac{1}{2!}(-ie)^2 \int d^4x_1 d^4x_2 T\Big(\bar{\psi}(x_1)\gamma_{\mu_1}\psi(x_1)A^{\mu_1}(x_1)\bar{\psi}(x_2)\gamma_{\mu_2}\psi(x_2)A^{\mu_2}(x_2)\Big).$$

Application of Wick's theorem yields contraction terms. We first note that contractions of the form

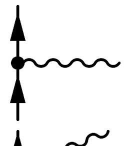
$$\underbrace{\psi(x_1)\psi}(x_2) \qquad \underbrace{\overline{\psi}(x_1)\overline{\psi}}(x_2)$$

vanish because they contain creators and annihilators, respectively, for different particles and thus

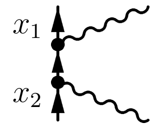
$$\langle 0|\, T(\psi(x_1)\psi(x_2))\, |0\rangle = 0.$$

The remaining terms read, using shorthands like $\bar{\psi}(x_1) = \bar{\psi}_1$,

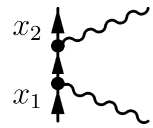
$\mathcal{S}^{(2)} = \frac{(-ie)^2}{2!}\int d^4x_1 d^4x_2 \big\{$



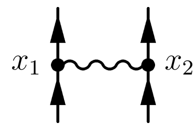
$:\bar\psi_1\gamma_{\mu_1}\psi_1\bar\psi_2\gamma_{\mu_2}\psi_2 A_1^{\mu_1}A_2^{\mu_2}:$ (a)



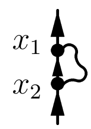
$+:\bar\psi_1\gamma_{\mu_1}\underbrace{\psi_1\bar\psi_2}\gamma_{\mu_2}\psi_2 A_1^{\mu_1}A_2^{\mu_2}:$ (b)



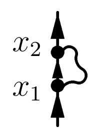
$+:\underbrace{\bar\psi_1\gamma_{\mu_1}\psi_1\bar\psi_2\gamma_{\mu_2}\psi_2} A_1^{\mu_1}A_2^{\mu_2}:$ (c)



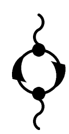
$+:\bar\psi_1\gamma_{\mu_1}\psi_1\bar\psi_2\gamma_{\mu_2}\psi_2 \underbrace{A_1^{\mu_1}A_2^{\mu_2}}:$ (d)



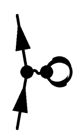
$+:\bar\psi_1\gamma_{\mu_1}\underbrace{\psi_1\bar\psi_2}\gamma_{\mu_2}\psi_2 \underbrace{A_1^{\mu_1}A_2^{\mu_2}}:$ (e)



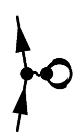
$+:\underbrace{\bar\psi_1\gamma_{\mu_1}\psi_1\bar\psi_2\gamma_{\mu_2}\psi_2} \underbrace{A_1^{\mu_1}A_2^{\mu_2}}:$ (f)



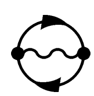
$+:\underbrace{\bar\psi_1\gamma_{\mu_1}\overbrace{\psi_1\bar\psi_2}\gamma_{\mu_2}\psi_2} A_1^{\mu_1}A_2^{\mu_2}:$ (g)



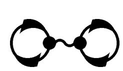
$+:\bar\psi_1\gamma_{\mu_1}\psi_1 \underbrace{\bar\psi_2\gamma_{\mu_2}\psi_2} \underbrace{A_1^{\mu_1}A_2^{\mu_2}}:$ (h)



$+:\underbrace{\bar\psi_1\gamma_{\mu_1}\psi_1} \bar\psi_2\gamma_{\mu_2}\psi_2 \underbrace{A_1^{\mu_1}A_2^{\mu_2}}:$ (i)



$+:\underbrace{\bar\psi_1\gamma_{\mu_1}\underbrace{\psi_1\bar\psi_2}\gamma_{\mu_2}\psi_2} \underbrace{A_1^{\mu_1}A_2^{\mu_2}}:$ (j)



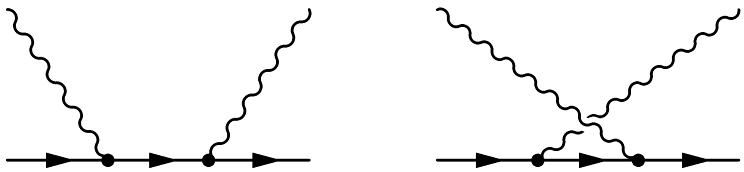
$+:\underbrace{\bar\psi_1\gamma_{\mu_1}\psi_1} \underbrace{\bar\psi_2\gamma_{\mu_2}\psi_2} \underbrace{A_1^{\mu_1}A_2^{\mu_2}}:\big\}.$ (k)

It follows a discussion of the contributions (a) through (k).

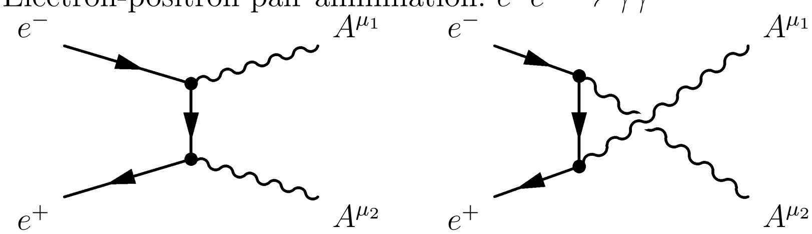
(a) *Independent emission or absorption.* These diagrams violate energy-momentum conservation.

(b)&(c) *Processes involving two electrons or positrons and two photons.*

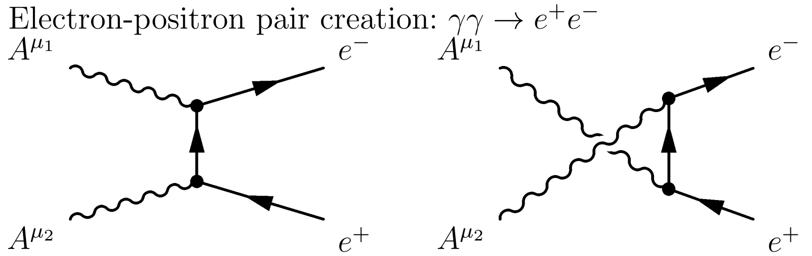
1. Compton scattering: $\gamma e^- \to \gamma e^-$, $\gamma e^+ \to \gamma e^+$



2. Electron-positron pair annihilation: $e^+e^- \to \gamma\gamma$

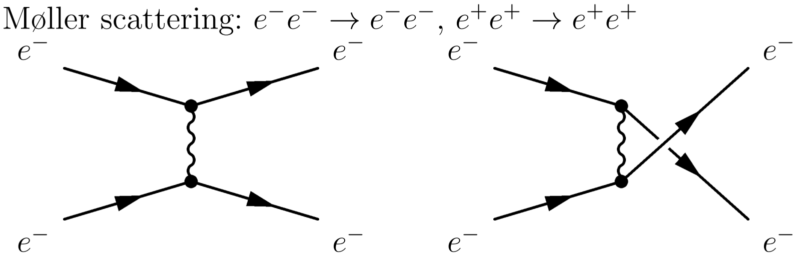


3. Electron-positron pair creation: $\gamma\gamma \to e^+e^-$

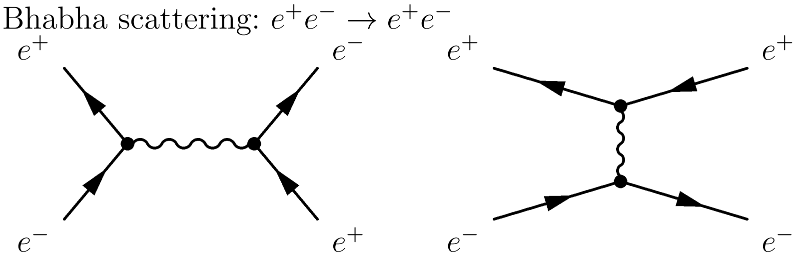


(d) *Processes involving four electrons or positions.*

1. Møller scattering: $e^-e^- \to e^-e^-$, $e^+e^+ \to e^+e^+$



2. Bhabha scattering: $e^+e^- \to e^+e^-$



(e)&(f) *No interaction between external particles.* No scattering takes place, these terms are corrections to the fermion propagator.

(g) *Correction to photon propagator.*

(h)&(i) *Corrections to fermion propagator, vanishing.*

(j)&(k) *Vacuum → vacuum transitions, disconnected graphs.*

This constitutes a list of all known processes (for practical purposes) in $\mathcal{S}^{(2)}$; in general, we can find all processes by examining all orders of the scattering matrix operator $\mathcal{S}$.

The $\mathcal{S}$-matrix elements are defined as matrix elements between single-particle states. Consequently, we need to apply the norm (5.49) repectively (5.50) to external states. The invariant amplitudes $\mathcal{M}_{fi}$, which are derived from the $\mathcal{S}$-matrix elements according to Eq. (3.11) properly account for this normalization factor, and are evaluated for continuum states as defined in Eq. (5.47) and Eq. (5.48).

The contractions of the field operators (see Eq. (5.43) and (5.44)) with external momentum eigenstates (as given in Eq. (5.49) and (5.50)) are for electrons

$$\begin{aligned}
\psi(x)\left|e^-(p,s)\right\rangle_{\text{single-particle}} &= \frac{1}{\sqrt{2E_pV}}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\sum_r a_r(k)u_r(k)\sqrt{2E_p}a_s^\dagger(p)\left|0\right\rangle \\
&= \frac{1}{\sqrt{2E_pV}}e^{-ipx}u_s(p)\left|0\right\rangle \\
\left\langle e^-(p,s)\right|_{\text{single-particle}}\bar{\psi}(x) &= \frac{1}{\sqrt{2E_pV}}e^{+ipx}\left\langle 0\right|\bar{u}_s(p),
\end{aligned}$$

for positrons

$$\begin{aligned}
\bar{\psi}(x)\left|e^+(p,s)\right\rangle_{\text{single-particle}} &= \frac{1}{\sqrt{2E_pV}}e^{-ipx}\bar{v}_s(p)\left|0\right\rangle \\
\left\langle e^+(p,s)\right|_{\text{single-particle}}\psi(x) &= \frac{1}{\sqrt{2E_pV}}e^{+ipx}\left\langle 0\right|v_s(p),
\end{aligned}$$

and for photons

$$\begin{aligned}
A_\mu(x)\left|\gamma(k,\lambda)\right\rangle &= \frac{1}{\sqrt{2E_kV}}e^{-ikx}\varepsilon_\mu^\lambda(k) \\
\left\langle\gamma(k,\lambda)\right|A_\mu(x) &= \frac{1}{\sqrt{2E_kV}}e^{+ikx}\varepsilon_\mu^{*\lambda}(k).
\end{aligned}$$

Example We treat the case of Møller scattering $e^-e^- \to e^-e^-$ as a typical example for the application of the Feynman rules.

We first define our initial and final states,

$$
\begin{aligned}
|i\rangle &= \left|e^-(p_1,s_1)\right\rangle_{\text{single-particle}} \otimes \left|e^-(p_2,s_2)\right\rangle_{\text{single-particle}} = \sqrt{2E_1 2E_2}\frac{1}{\sqrt{2E_1V2E_2V}}a^\dagger_{s_1}(p_1)a^\dagger_{s_2}(p_2)\,|0\rangle\,, \\
\langle f| &= \left\langle e^-(p_3,s_3)\right|_{\text{single-particle}} \otimes \left\langle e^-(p_4,s_4)\right|_{\text{single-particle}} = \sqrt{2E_3 2E_4}\frac{1}{\sqrt{2E_3V2E_4V}}\,\langle 0|\, a_{s_4}(p_4)a_{s_3}(p_3).
\end{aligned}
$$

The transition matrix element $\mathcal{S}_{fi}$ is then,

$$
\begin{aligned}
\mathcal{S}_{fi} = \langle f|\,\mathcal{S}\,|i\rangle = \frac{(-ie)^2}{2!}\int d^4x_1 d^4x_2 \sqrt{16E_1E_2E_3E_4}\,\langle 0|\,\underbrace{a_{s_4}(p_4)}_{E}\,\underbrace{a_{s_3}(p_3)}_{D} \\
:\underbrace{\bar\psi(x_1)}_{D}\,\gamma_\mu\,\underbrace{\underbrace{\psi(x_1)}_{C}\,\underbrace{\bar\psi(x_2)}_{E}}_{\overset{B}{=}-\bar\psi(x_2)\psi(x_1)}\,\gamma_\nu\,\underbrace{\psi(x_2)}_{A}:\,A^\mu(x_1)A^\nu(x_2)\,\underbrace{a^\dagger_{s_1}(p_1)}_{A}\,\underbrace{a^\dagger_{s_2}(p_2)}_{C}\,|0\rangle\,,
\end{aligned}
\tag{5.95}
$$

where $A^\mu(x_1)$ and $A^\nu(x_2)$ are contracted.

yielding $2\times 2 = 4$ Feynman graphs in position space (of which 2! are topologically identical). In Fig. 5.7, we labeled the last Feynman graph according to Eq. (5.95).

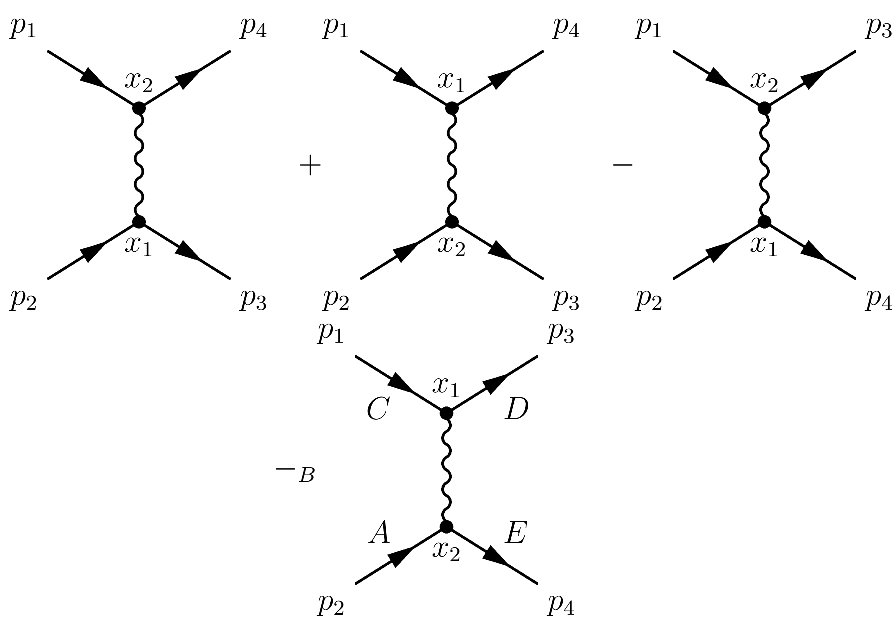


Figure 5.7: Feynman graphs associated with the Møller scattering.

We recall that each ordering of ψ, $\bar\psi$ corresponds to a Feynman diagram. The anticommutation relations are responsible for the relative sign changes.

With the photon propagator in momentum space,

$$iD_F^{\mu\nu}(q) = -\frac{ig^{\mu\nu}}{q^2}, \tag{5.96}$$

we get,

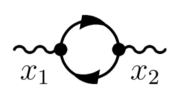
$$\begin{aligned}\mathcal{S}_{fi} =& (-ie)^2(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)\frac{1}{\sqrt{16E_1E_2E_3E_4V^2}}\\ &\Big[\bar{u}_{s_4}(p_4)\gamma_\mu u_{s_2}(p_2)iD_F^{\mu\nu}(p_3-p_1)\bar{u}_{s_3}(p_3)\gamma_\mu u_{s_1}(p_1)\\ &-\bar{u}_{s_4}(p_4)\gamma_\mu u_{s_1}(p_1)iD_F^{\mu\nu}(p_3-p_2)\bar{u}_{s_3}(p_3)\gamma_\mu u_{s_2}(p_2)\Big].\end{aligned} \tag{5.97}$$

We now define the **invariant amplitude** $\mathcal{M}_{fi}$ (see Eq. (3.11)) via,

$$\mathcal{S}_{fi} = \delta_{fi} + i(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)\frac{1}{\sqrt{16E_1E_2E_3E_4V^2}}\mathcal{M}_{fi}. \tag{5.98}$$

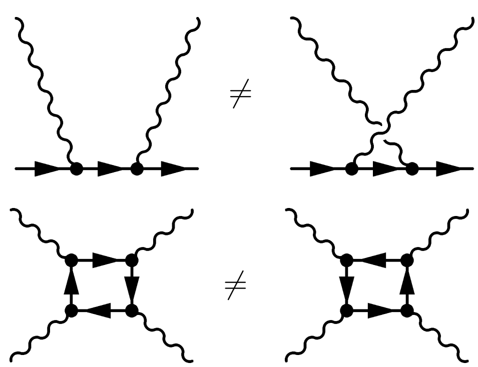
$\mathcal{M}_{fi}$ can then be computed using the Feynman rules in momentum space.

Application of the Feynman rules

- Momentum conservation at each vertex
- Fermion number conservation at each vertex (indicated by the direction of the arrows)
- All topologically allowed graphs contribute
- Exchange factor (-1) when interchanging two external fermions with each other
- Each closed fermion loop yields a factor (-1), e.g.  coming from the contraction $:\overline{\psi}(x_1)\psi(x_1)\overline{\psi}(x_2)\psi(x_2):$
- graphs in which the ordering of the vertices along a fermion line is different are not topologically equivalent, and must be summed, eg.

$\mathcal{S}^{(2)}$: $\neq$

$\mathcal{S}^{(4)}$: $\neq$



External states		
incoming electron	p, s	$u_s(p)$
outgoing electron	p, s	$\bar{u}_s(p)$
incoming positron	p, s	$\bar{v}_s(p)$
outgoing positron	p, s	$v_s(p)$
incoming photon	λ, μ, p	$(\varepsilon_\lambda)^\mu(p)$
outgoing photon	λ, μ, p	$(\varepsilon^*_\lambda)^\mu(p)$
Propagators		
electron	p	$\frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$
photon	μ, ν, k	$\frac{-ig^{\mu\nu}}{k^2+i\varepsilon}$
Vertex		
electron-photon-electron	μ, q_e	$-ieq_e\gamma^\mu$

Table 5.2: Feynman rules in momentum space.

5.9 Trace techniques for γ-matrices

Cross sections are proportinal to $|\mathcal{M}_{fi}|^2 \propto |\bar{u}_{s_f}(p_f)\Gamma u_{s_i}(p_i)|^2$, where Γ denotes an arbitrary product of γ-matrices.

In many experiments – but not all! –, the spin states of the initial and final states are not observed. This is for example the case at the CMS and ATLAS experiments of LHC. We then need to follow the following procedure :

- If the spin state of the final state particles cannot be measured, one must sum over the final state spins : $\sum_{s_f} |\cdots|^2$,
- If the initial states particles are unpolarized, one must average over the initial state spins : $\frac{1}{2}\sum_{s_i} |\cdots|^2$.

Then, remembering that $\bar{u} = u^\dagger \gamma^0$, we can write,

$$
\begin{aligned}
\frac{1}{2}\sum_{s_i,s_f} |\bar{u}_{s_f}(p_f)\Gamma u_{s_i}(p_i)|^2 &= \frac{1}{2}\sum_{s_i,s_f} \bar{u}_{s_f}(p_f)\Gamma u_{s_i}(p_i) u^\dagger_{s_i}(p_i)\gamma^0\gamma^0\Gamma^\dagger\gamma^0 u_{s_f}(p_f) \\
&= \frac{1}{2}\sum_{s_i,s_f} \bar{u}_{s_f}(p_f)\Gamma u_{s_i}(p_i)\bar{u}_{s_i}(p_i)\bar{\Gamma} u_{s_f}(p_f) \\
&= \frac{1}{2}\sum_{s_i,s_f} (\bar{u}_{s_f}(p_f))_\alpha \Gamma_{\alpha\beta}(u_{s_i}(p_i))_\beta (\bar{u}_{s_i}(p_i))_\gamma \bar{\Gamma}_{\gamma\delta}(u_{s_f}(p_f))_\delta \\
&\overset{(5.41)}{=} \frac{1}{2}\Gamma_{\alpha\beta}(\not{p}_i + m)_{\beta\gamma}\bar{\Gamma}_{\gamma\delta}(\not{p}_f + m)_{\delta\alpha} \\
&= \frac{1}{2}\Big(\Gamma(\not{p}_i + m)\bar{\Gamma}(\not{p}_f + m)\Big)_{\alpha\alpha} \\
&= \frac{1}{2}\mathrm{Tr}\Big(\Gamma(\not{p}_i + m)\bar{\Gamma}(\not{p}_f + m)\Big),
\end{aligned}
$$

where the indices α, β, γ and δ label the matrix element, and $\bar{\Gamma} := \gamma^0\Gamma^\dagger\gamma^0$.

We thus get the important result,

$$
\boxed{\frac{1}{2}\sum_{s_i,s_f} |\bar{u}_{s_f}(p_f)\Gamma u_{s_i}(p_i)|^2 = \frac{1}{2}\mathrm{Tr}\Big(\Gamma(\not{p}_i + m)\bar{\Gamma}(\not{p}_f + m)\Big)}, \tag{5.99}
$$

and its analogon for antiparticles,

$$
\boxed{\frac{1}{2}\sum_{s_i,s_f} |\bar{v}_{s_f}(p_f)\Gamma v_{s_i}(p_i)|^2 = \frac{1}{2}\mathrm{Tr}\Big(\Gamma(\not{p}_i - m)\bar{\Gamma}(\not{p}_f - m)\Big)}, \tag{5.100}
$$

i.e. the Clifford algebra of γ-matrices is taking care of the spin summation for us.

We now compute $\bar{\Gamma}$ for an arbitrary number of γ-matrices.

- For $\Gamma = \gamma^\mu$, $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma_i$ hence $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \Rightarrow \bar{\gamma^\mu} = \gamma^\mu$. For later use, note that $\bar{\gamma^5} = -\gamma^5$.
- For $\Gamma = \gamma^{\mu_1}\cdots\gamma^{\mu_n}$, $\Gamma^\dagger = (\gamma^{\mu_1}\cdots\gamma^{\mu_n})^\dagger = \gamma^0\gamma^{\mu_n}\cdots\gamma^{\mu_1}\gamma^0 \Rightarrow \bar{\Gamma} = \gamma^{\mu_n}\cdots\gamma^{\mu_1}$. In other words, to get $\bar{\Gamma}$, we just need to read Γ in the inverse ordering.

We finally want to compute some traces for products of γ-matrices, since they appear explicitly $(\Gamma, \bar{\Gamma})$ and implicitely ($\not{p} = \gamma^\mu p_\mu$) in the formulas (5.99) and (5.100). In doing this, one should remember that the trace is cyclic ($\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$) and the Clifford algebra of γ-matrices.

- **0 γ-matrix :** $\mathrm{Tr}\,\mathbb{1} = 4$.
- **1 γ-matrix :** $\mathrm{Tr}\,\gamma^\mu = 0$, $\mathrm{Tr}\,\gamma^5 = 0$. The last one is shown using the fact that $\{\gamma^5, \gamma^\mu\} = 0$.
- **2 γ-matrices :** $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = 4g^{\mu\nu} \Rightarrow \mathrm{Tr}(\not{a}\not{b}) = 4a\cdot b$, where $\cdot$ is the scalar product of 4-vectors.
- **4 γ-matrices :**

$$\begin{aligned}
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu) = -\mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma) + 2g^{\mu\sigma}\mathrm{Tr}(\gamma^\nu\gamma^\rho) \\
&= \mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma) + 8g^{\mu\sigma}g^{\nu\rho} - 8g^{\mu\rho}g^{\nu\sigma} \\
&= -\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) + 8g^{\mu\sigma}g^{\nu\rho} - 8g^{\mu\rho}g^{\nu\sigma} + 8g^{\mu\nu}g^{\rho\sigma} \\
\Rightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}) \\
\Rightarrow \mathrm{Tr}(\not{a}_1\not{a}_2\not{a}_3\not{a}_4) &= 4\left[(a_1\cdot a_2)(a_3\cdot a_4) + (a_1\cdot a_4)(a_2\cdot a_3) - (a_1\cdot a_3)(a_2\cdot a_4)\right].
\end{aligned}$$

  and in general,

$$\begin{aligned}
\mathrm{Tr}(\not{a}_1\cdots\not{a}_n) =&(a_1\cdot a_2)\mathrm{Tr}(\not{a}_3\cdots\not{a}_n) - (a_1\cdot a_3)\mathrm{Tr}(\not{a}_3\not{a}_4\cdots\not{a}_n) \\
&+ \cdots \pm (a_1\cdot a_n)\mathrm{Tr}(\not{a}_2\cdots\not{a}_{n-1}),
\end{aligned}$$

  which implies inductivly that the trace of a string of γ-matrices is a real number.

- **n γ-matrices (n odd) :**

$$\begin{aligned}
\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) &= \mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}\underbrace{\gamma^5\gamma^5}_{=\mathbb{1}}) = \mathrm{Tr}(\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_n}\gamma^5) \\
&= (-1)^n\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) \Rightarrow \mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = 0.
\end{aligned}$$

- **n γ-matrices (n even) :**

$$\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = \mathrm{Tr}((\gamma^{\mu_1}\cdots\gamma^{\mu_n})^\dagger) = \mathrm{Tr}(\gamma^0\gamma^{\mu_n}\cdots\gamma^{\mu_1}\gamma^0) = \mathrm{Tr}(\gamma^{\mu_n}\cdots\gamma^{\mu_1}).$$

- γ^5 **and 2** γ**-matrices :** $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$. To show this identity, we remark that $\gamma^5\gamma^\mu\gamma^\nu$ is a rank-2 tensor, which does not depend on any 4-momenta. Therefore, $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = cg^{\mu\nu}$. We contract with $g_{\mu\nu}$ to get $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma_\mu) = cg^{\mu\nu}g_{\mu\nu} = 4c$, but since $\gamma^\mu\gamma_\mu = 4\mathbb{1}$ we get $c = \mathrm{Tr}\,\gamma^5 = 0$.
- γ^5 **and 4** γ**-matrices :** $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\varepsilon^{\mu\nu\rho\sigma}$.
- **Contractions :**

$$\gamma^\mu\gamma_\mu = 4\mathbb{1} \tag{5.101}$$
$$\gamma^\mu \not{a}\gamma_\mu = -2\not{a} \tag{5.102}$$
$$\gamma^\mu \not{a}\not{b}\gamma_\mu = 4(a\cdot b)\mathbb{1} \tag{5.103}$$
$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a} \tag{5.104}$$

5.10 Annihilation process : $e^+e^- \to \mu^+\mu^-$

In this section, we compute the differential cross section of the simplest of all QED process, the reaction

$$e^-(p_1)e^+(p_2) \to \mu^-(p_3)\mu^+(p_4),$$

illustrated on Fig. 5.8. The simplicity arises frome the fact that $e^- \neq \mu^-$, and hence only one diagram contributes (the e^+e^--pair *must* be annihilated).

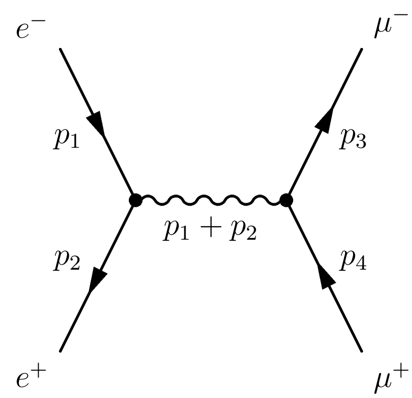


Figure 5.8: Annihilation process $e^+e^- \to \mu^+\mu^-$

We recall the Mandelstam variables for this process,

$$s = (p_1+p_2)^2$$
$$t = (p_1-p_3)^2$$
$$u = (p_1-p_4)^2.$$

We make the following assumptions (very common for QED processes),

- Unpolarized leptons : $\frac{1}{2}\sum_{s_i,s_f}$,
- High energy limit : $m_e, m_\mu = 0 \Leftrightarrow \sqrt{s} \gg m_e, m_\mu$.

Using the Feynman rules of Table 5.2 for the diagram depicted in Fig. 5.8, we get,

$$\begin{aligned}
-i\mathcal{M}_{fi} &= \bar{u}_{s_3}(p_3)ie\gamma^\mu v_{s_4}(p_4)\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\bar{v}_{s_2}(p_2)ie\gamma^\nu u_{s_1}(p_1) \\
\overline{|\mathcal{M}_{fi}|^2} &= \frac{1}{2}\sum_{s_1}\frac{1}{2}\sum_{s_2}\sum_{s_3}\sum_{s_4}|\mathcal{M}_{fi}|^2 \\
&= \frac{1}{4}\frac{e^4}{s^2}\mathrm{Tr}(\gamma^\mu \not{p}_4 \gamma^\nu \not{p}_3)\mathrm{Tr}(\gamma_\mu \not{p}_1 \gamma_\nu \not{p}_2) \\
&= \frac{1}{4}\frac{e^4}{s^2}16\left[2(p_1\cdot p_3)(p_2\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3)\right].
\end{aligned}$$

Since we are working in the high energy limit, we have $p_i^2 = 0$ and hence $t = -2p_1 \cdot p_3 = -2p_2 \cdot p_4$ and $u = -2p_1 \cdot p_4 = -2p_2 \cdot p_3$. Using the identity $s + t + u = 2m_e^2 + 2m_\mu^2 = 0$ to get rid of the Mandelstam u-variable and with $\alpha = \frac{e^2}{4\pi}$ we have,

$$\overline{|\mathcal{M}_{fi}|^2} = 32\pi^2\alpha^2\frac{t^2 + (s+t)^2}{s^2}. \tag{5.105}$$

Considering the center of mass frame, we have $s = 4(E^*)^2$, $t = -\frac{s}{2}(1 - \cos\Theta^*)$ and with the help of Eq. (3.34), this yields,

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\overline{|\mathcal{M}_{fi}|^2} = \frac{\pi\alpha^2}{s^2}(1 + \cos^2\Theta^*), \tag{5.106}$$

or using,

$$\frac{d\sigma}{dt} = \frac{d\Omega^*}{dt}\frac{d\sigma}{d\Omega^*} = \frac{4\pi}{s}\frac{d\sigma}{d\Omega^*},$$

we get the differential cross section for $e^+e^- \to \mu^+\mu^-$ in the center of mass frame,

$$\boxed{\frac{d\sigma^{e^+e^-\to\mu^+\mu^-}}{d\Omega^*} = \frac{\alpha^2}{4s}(1 + \cos^2\Theta^*)}. \tag{5.107}$$

This differential cross section (see Fig. 5.9) has been very well measured and is one of the best tests of QED at high energies.

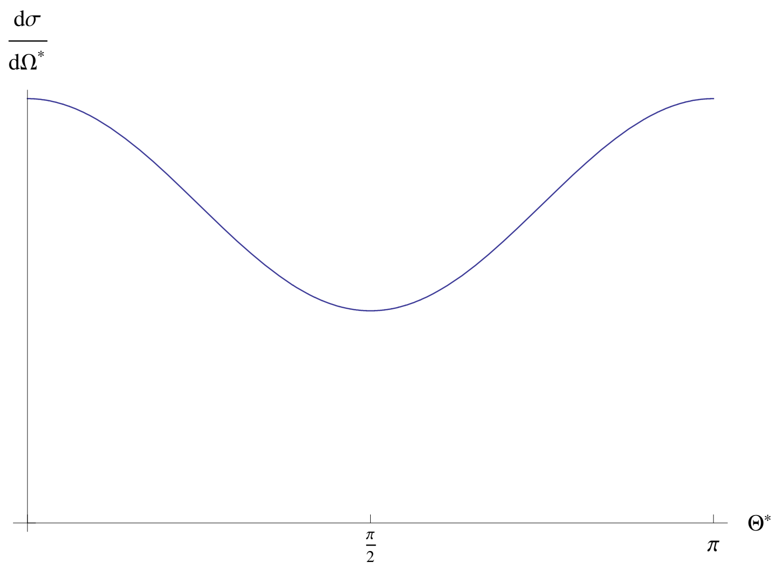


Figure 5.9: Differential cross section for $e^+e^- \to \mu^+\mu^-$ in the center of mass frame.

Using this result, we can calculate the total cross-section by integration over the solid angle:

$$\sigma = \int \frac{d\sigma}{d\Omega^*} d\Omega^* = \frac{\alpha^2}{4s} \int_0^\pi (1+\cos^2\Theta^*) \underbrace{\sin\Theta^* d\Theta^*}_{d\cos\Theta^*} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \tag{5.108}$$

$$= \frac{\alpha^2}{4s} 2\pi \frac{8}{3} \tag{5.109}$$

$$\Rightarrow \boxed{\sigma^{e^+e^-\to\mu^+\mu^-} = \frac{4\pi\alpha^2}{3s}} = \frac{86.9\,\text{nb}}{s\,[\text{GeV}^2]} \tag{5.110}$$

where $1\,\text{nb} = 10^{-33}\,\text{cm}^2$. If one considers non-asymptotic energies, $s \simeq m_\mu^2$ (but $s \gg m_e^2$), one finds a result which reduces to Eq. (5.110) for $m_\mu^2 = 0$:

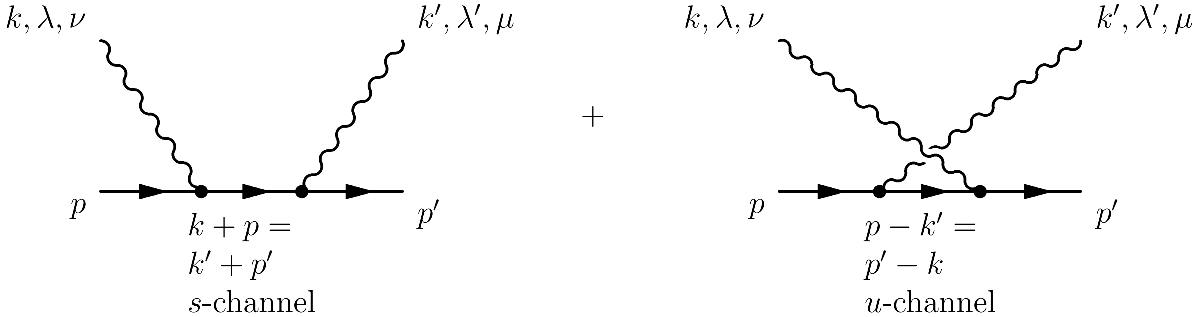
$$\sigma^{e^+e^-\to\mu^+\mu^-} = \frac{4\pi\alpha^2}{3s}\left(1 + 2\frac{m_\mu^2}{s}\right)\sqrt{1 - \frac{4m_\mu^2}{s}}.$$

5.11 Compton scattering

Let us now consider Compton scattering:

$$\gamma(k) + e^-(p) \to \gamma(k') + e^-(p').$$

The diagrams corresponding to this process have been introduced in Sect. 5.8.



This yields the amplitude

$$-i\mathcal{M}_{fi} = \varepsilon^*_\mu(k', \lambda')\varepsilon_\nu(k, \lambda)\bar{u}(p') \left[\underbrace{ie\gamma^\mu \frac{i}{\not{p} + \not{k} - m} ie\gamma^\nu}_{\text{LHS diagram}} + \underbrace{ie\gamma^\nu \frac{i}{\not{p} - \not{k}' - m} ie\gamma^\mu}_{\text{RHS diagram}} \right] u(p)$$

where the on-shell conditions read

$$k^2 = k'^2 = 0 \qquad\qquad p^2 = p'^2 = m^2$$

and the photons are transversal:

$$k \cdot \varepsilon(k) = k' \cdot \varepsilon(k') = 0.$$

It is instructive to check that the invariant amplitude is indeed also gauge invariant. Consider the gauge transformation

$$A_\nu(x) \to A_\nu(x) + \partial_\nu \Lambda(x)$$

which leaves Maxwell's equations unaltered. In the photon field operator this can be implemented by

$$\varepsilon_\nu(k, \lambda) \to \varepsilon_\nu(k, \lambda) + \beta k_\nu, \;\; \beta \in \mathbb{R} \text{ arbitrary.}$$

We observe the change of the matrix element for transformation of one of the photons:

$$-i\mathcal{M}_{fi}(\varepsilon_\nu \to k_\nu) = -ie^2 \varepsilon^*_\mu(k', \lambda')\bar{u}(p') \left[\gamma^\mu \frac{1}{\not{p} + \not{k} - m} \not{k} + \not{k} \frac{1}{\not{p} - \not{k}' - m} \gamma^\mu \right] u(p).$$

In simplifying this expression, we use

$$\frac{1}{\not{p} + \not{k} - m} \not{k} u(p) = \frac{1}{\not{p} + \not{k} - m} (\not{k} + \not{p} - m) u(p) = \mathbb{1} u(p)$$

where we added a zero since $(\not p - m)u(p) = 0$ and analogously

$$\bar{u}(p')\not k \frac{1}{\not p - \not k' - m} = \bar{u}(p')(\not k - \not p' + m)\frac{1}{\not p' - \not k - m} = -\bar{u}(p')\mathbb{1}.$$

Putting the terms together, we therefore find

$$-i\mathcal{M}_{fi}(\varepsilon_\nu \to k_\nu) = -ie^2\varepsilon^*_\mu(k',\lambda')\bar{u}(p')(\gamma^\mu\mathbb{1} - \mathbb{1}\gamma^\mu)u(p) = 0.$$

The result is the same for the transformation $\varepsilon^*_\mu \to \varepsilon^*_\mu + \beta k'_\mu$.

It is generally true that only the sum of the contributing diagrams is gauge invariant. Individual diagrams are not gauge invariant and thus without physical meaning.

Recall that the aim is to find the differential cross section and therefore the squared matrix element. Since there are two contributing diagrams, one has to watch out for interference terms. Applying the trace technology developed in Sect. 5.9 yields

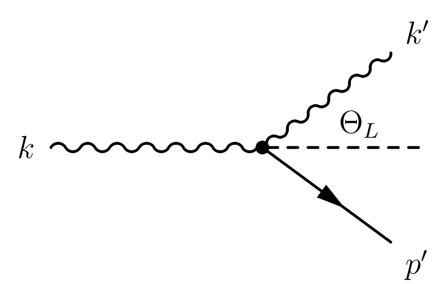
$$\begin{aligned}\overline{|\mathcal{M}_{fi}|^2} &= \frac{1}{2}\sum_\lambda \frac{1}{2}\sum_s \sum_{\lambda'} \sum_{s'} |\mathcal{M}_{fi}|^2 \\ &= 2e^4\left[\frac{m^2-u}{s-m^2} + \frac{m^2-s}{u-m^2} + 4\left(\frac{m^2}{s-m^2} + \frac{m^2}{u-m^2}\right) + 4\left(\frac{m^2}{s-m^2} + \frac{m^2}{u-m^2}\right)^2\right].\end{aligned} \tag{5.111}$$

Bearing in mind that $s + t + u = 2m^2$, this yields the unpolarized Compton cross-section

$$\frac{d\sigma}{dt} = \frac{1}{16\pi(s-m^2)^2}\overline{|\mathcal{M}_{fi}|^2} \tag{5.112}$$

which is a frame independent statement.

Head-on electron-photon collision is rather uncommon; usually photons are hitting on a target. Therefore it is useful to consider the electron's rest frame (laboratory frame):



With $\omega = |\vec{k}| = E^L_\gamma$, $\omega' = |\vec{k}'| = E'^L_\gamma$, and $p = (m, \vec{0})^T$ one finds

$$s - m^2 = 2m\omega \tag{5.113}$$

$$u - m^2 = -2p\cdot k' = -2m\omega' \tag{5.114}$$

$$t = -2\omega\omega'(1 - \cos\Theta_L). \tag{5.115}$$

One of the three variables can be eliminated using $s + t + u = 2m^2$:

$$\omega' = \frac{1}{2m}(s + t - m^2) = \omega - \frac{\omega\omega'}{m}(1 - \cos\Theta_L) \tag{5.116}$$

$$\Rightarrow \frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1 - \cos\Theta_L) \tag{5.117}$$

$$\Rightarrow \omega' = \frac{\omega}{1 + \frac{\omega}{m}(1 - \cos\Theta_L)}. \tag{5.118}$$

We continue calculating the differential cross-section. Eq. (5.115) yields

$$dt = \frac{\omega'^2}{\pi} 2\pi d\cos\Theta_L = \frac{\omega'^2}{\pi} d\Omega_L.$$

Furthermore, we can use Eq. (5.117) to simplify Eq. (5.111):

$$\frac{m^2}{s - m^2} + \frac{m^2}{u - m^2} = \frac{m^2}{2m\omega} + \frac{m^2}{-2m\omega'} = \frac{m}{2}\left(\frac{1}{\omega} - \frac{1}{\omega'}\right) = -\frac{1}{2}(1 - \cos\Theta_L).$$

Using these results and remembering Eq. (5.112), we obtain

$$\frac{d\sigma^{\gamma e \to \gamma e}}{d\Omega_L} = \frac{dt}{d\Omega_L}\frac{d\sigma}{dt} = \frac{\omega'^2}{\pi}\frac{1}{16\pi(2m\omega)^2} 2e^2 \left[\frac{2m\omega'}{2m\omega} + \frac{-2m\omega}{-2m\omega'} - \sin^2\Theta_L\right] \tag{5.119}$$

$$= \frac{\omega'^2}{\pi}\frac{2 \cdot 16\pi^2\alpha^2}{16\pi 4m^2\omega^2}\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\Theta_L\right] \tag{5.120}$$

$$\Rightarrow \boxed{\frac{d\sigma^{\gamma e \to \gamma e}}{d\Omega_L} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\Theta_L\right]} \tag{5.121}$$

which is called the Klein-Nishima formula.

It follows a discussion of important limiting cases.

- *Classical limit:* $\omega \ll m \Rightarrow \omega' \simeq \omega$
  In the classical limit, Eq. (5.121) simplifies to the classical Thomson cross-section (which was used to measure α)

  $$\boxed{\frac{d\sigma^{\gamma e \to \gamma e}}{d\Omega_L} = \frac{\alpha^2}{2m^2}\left[1 + \cos^2\Theta_L\right]},$$

  yielding the total cross-section

  $$\boxed{\sigma^{\gamma e \to \gamma e} = \frac{\alpha^2}{2m^2}\frac{16\pi}{3}}.$$

- *Asymptotic limit:* $s \gg m^2 \Rightarrow \omega \gg m$
  In this case, the so-called leading log approximation holds:
$$\sigma^{\gamma e \to \gamma e} = \frac{2\pi\alpha^2}{m^2}\frac{m^2}{s}\left[\ln\frac{s}{m^2} + \frac{1}{2} + \mathcal{O}\left(\frac{m^2}{s}\right)\right] \simeq \frac{2\pi\alpha^2}{s}\ln\frac{s}{m^2}.$$

- *In general* we can conclude that
$$\sigma^{\gamma e \to \gamma e} \sim \frac{\alpha^2}{m^2} \simeq 10^{-25}\,\mathrm{cm}^2$$
  from which one can infer the "classical electron radius"
$$r_e^{\text{classical}} \sim \sqrt{\sigma_{\text{Thomson}}} \sim \frac{\alpha}{m} = 2.8 \cdot 10^{-13}\,\mathrm{cm}.$$

5.12 QED as a gauge theory

Recall the QED Lagrangian
$$\begin{aligned}\mathcal{L}^{\text{QED}} &= \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eq_e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \mathcal{L}_0^{\text{Dirac}} + \mathcal{L}' + \mathcal{L}_0^{\text{photon}}\end{aligned}$$
introduced in Sect. 5.8 which includes the following observables:

- *Fermions:* components of $\bar{\psi}\gamma^\mu\psi = j^\mu$
- *Photons:* components of $F^{\mu\nu}$: $\vec{E}$ and $\vec{B}$ field.

Neither ψ nor A_μ as such are observables. In particular, the phase of ψ cannot be observed. This means that QED must be invariant under phase transformations of ψ:
$$\psi(x) \to \psi'(x) = e^{ieq_e\chi(x)}\psi(x)$$
which is a unitary one-dimensional i. e. $U(1)$ transformation. Observe first the action on the Dirac Lagrangian:
$$\begin{aligned}\mathcal{L}_0^{\text{Dirac}} &\to \bar{\psi}'(i\gamma^\mu\partial_\mu - m)\psi' \\ &= \bar{\psi}e^{-ieq_e\chi(x)}e^{ieq_e\chi(x)}(i\partial\!\!\!/ - m)\psi - \bar{\psi}\gamma^\mu(\partial_\mu eq_e\chi(x))\psi \\ &= \mathcal{L}_0^{\text{Dirac}} - eq_e\bar{\psi}\gamma^\mu\psi(\partial_\mu\chi(x)).\end{aligned}$$
Therefore, the free Dirac field Lagrangian alone is not invariant under this transformation. In order for the extra term to vanish, A_μ has to be transformed, too:
$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\chi(x)$$

QED	photon	$U(1)$
Weak interaction	$W^{\pm},\ Z^0$	$SU(2)$
QCD	gluon	$SU(3)$

Table 5.3: Summary of gauge theories.

such that $F^{\mu\nu} = F'^{\mu\nu}$ since $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. This means that we are dealing with the gauge transformation known from classical electrodynamics. Because we have

$$-eq_e\bar{\psi}\gamma^\mu\psi A_\mu \to -eq_e\bar{\psi}\gamma^\mu\psi A_\mu + eq_e\bar{\psi}\gamma^\mu\psi(\partial_\mu\chi)$$

the complete Lagrangian $\mathcal{L}^{\text{QED}}$ is invariant under $U(1)$ gauge transformations. This motivates the definition of the gauge covariant derivative

$$\boxed{D_\mu = \partial_\mu + ieq_eA_\mu}$$

which contains the photon-electron interaction.

In summary, the requirement of gauge invariance uniquely determines the photon-electron interaction and QED is a $U(1)$ gauge theory.

This suggests a new approach on theory building: start from symmetries instead of finding them in the final Lagrangian:

local symmeties (gauge invariance)	→	existence of vector fields (gauge fields)	→	gauge interactions

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A summary of gauge theories with the corresponding gauge fields and gauge groups is given in Tab. 5.3.