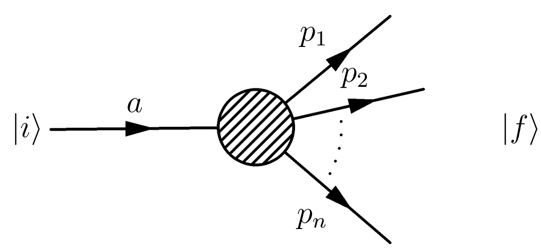
Chapter 3

Lorentz invariant scattering cross section and phase space

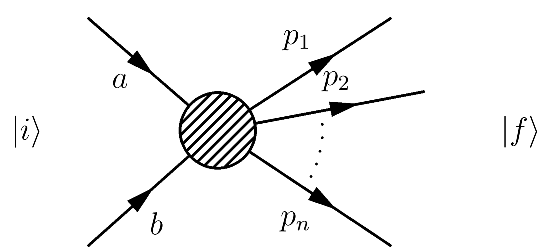
In particle physics, there are basically two observable quantities :

- Decay rates,
- Scattering cross-sections.

Decay:



Scattering:



3.1 $\mathcal{S}$-operator

In both cases $|i\rangle$ denotes the initial state, $|f\rangle$ denotes a multiparticle final state in a Fock space and the box represents the dynamics/interactions and is called the **$\mathcal{S}$-operator**. The last is predicted by the theory describing the interaction.

Example In QM I/II, $\mathcal{S} \propto H'(t) \propto V(t)$ in the first order perturbation theory of the Schrödinger equation.

$\mathcal{S}$ is usually a very complicated object : it contains the information about *all* possible transitions $|i\rangle \to |f\rangle$. Another way to state this is to remark that $\mathcal{S}$ contains all the dynamics of the process.

In experiments one does not get/need/want the full $\mathcal{S}$-operator. Instead, one restricts oneself to specific $|i\rangle$ and $|f\rangle$ e.g. by choosing the beam particles (muon beam,...) for the first and looking only at specific outcomes (3-jets events,...) for the latter.

One represents the $\mathcal{S}$-operator by looking at its matrix elements,

$$\underbrace{\sum_{f'} |f'\rangle \langle f'|}_{\mathbb{1}} \mathcal{S} |i\rangle = \sum_{f'} |f'\rangle \mathcal{S}_{f'i} \tag{3.1}$$

where

$$\mathcal{S}_{f'i} = \langle f'| \mathcal{S} |i\rangle \tag{3.2}$$

To isolate a specific outcome $|f\rangle$, one multiplies Eq. (3.1) by $\langle f|$, and gets,

$$\langle f| \sum_{f'} |f'\rangle \mathcal{S}_{f'i} = \sum_{f'} \underbrace{\langle f\,|f'\rangle}_{=\delta_{ff'}} \mathcal{S}_{f'i} = \mathcal{S}_{fi}. \tag{3.3}$$

Hence, the probability for the process $|i\rangle \to |f\rangle$ is,

$$P(|i\rangle \to |f\rangle) = |\mathcal{S}_{fi}|^2 \tag{3.4}$$

In general we can write,

$$\mathcal{S}_{fi} = \underbrace{\delta_{fi}}_{\text{no int.}} + \underbrace{i(2\pi)^4 \delta^{(4)}(p_f - p_i)}_{\text{4-momentum cons.}} \cdot \underbrace{\mathcal{T}_{fi}}_{\text{scat. amplitude}}, \tag{3.5}$$

or using a shorthand notation

$$\mathcal{S} = \mathbb{1} + i\mathcal{T},$$

in Feynman diagrams:

In the discussion of particle physics, a frequently used quantity is the transition probability per unit time,

$$w_{fi} = \frac{|\mathcal{S}_{fi}|^2}{T}. \tag{3.6}$$

3.2 Fermi's golden rule

From Eqs. (3.4) and (3.5), we see that we must address the issue of defining the value of a squared Dirac δ-function. To do this we use the rather pragmatic approach due to Fermi:

$$\begin{aligned}
\left[2\pi\delta(p_f^0 - p_i^0)\right]^2 &= \int dt\, \mathrm{e}^{i(p_f^0 - p_i^0)t} \cdot 2\pi\delta(p_f^0 - p_i^0) \\
&= T \cdot 2\pi\delta(p_f^0 - p_i^0) \qquad (3.7)\\
\left[(2\pi)^3\delta^{(3)}(\vec{p}_f - \vec{p}_i)\right]^2 &= \iiint d^3x\, \mathrm{e}^{i(\vec{p}_f - \vec{p}_i)\cdot\vec{x}} \cdot (2\pi)^3\delta^{(3)}(\vec{p}_f - \vec{p}_i) \\
&= V \cdot (2\pi)^3\delta^{(3)}(\vec{p}_f - \vec{p}_i) \qquad (3.8)\\
\Rightarrow w_{fi} &= \frac{|\mathcal{S}_{fi}|^2}{T} = V \cdot (2\pi)^4\delta^{(4)}(p_f - p_i) \cdot |\mathcal{T}_{fi}|^2 \qquad (3.9)
\end{aligned}$$

To talk about the transition rate, we look at a Fock-space with a fixed number of particles.

Experimentally, the angle and energy-momentum is only accessible up to a given accuracy. We therefore use differential cross-sections in angle $d\Omega$ and energy-momentum dp near Ω, p respectively.

Motivating example In a cubic box of volume $V = L^3$ with infinitely high potential wells, the authorized momentum-values are discretely distributed.

$$p = \frac{2\pi}{L} n \Rightarrow dn = \frac{L}{2\pi} dp \Rightarrow d^3n = \left(\frac{L}{2\pi}\right)^3 d^3p,$$

and hence,

$$dw_{fi} = V \cdot (2\pi)^4\delta^{(4)}(p_f - p_i) \cdot |\mathcal{T}_{fi}|^2 \cdot \prod_{f=1}^{n_f} \frac{V}{(2\pi)^3} d^3p_f, \tag{3.10}$$

where n_f stands for the number of particles in the final state.

In order to get rid of normalization factors, we define a new matrix element $\mathcal{M}_{fi}$ by,

$$\mathcal{T}_{fi} \stackrel{!}{=} \left(\prod_{i=1}^{n_i} \frac{1}{\sqrt{2E_i V}}\right)\left(\prod_{f=1}^{n_f} \frac{1}{\sqrt{2E_f V}}\right)\mathcal{M}_{fi}. \tag{3.11}$$

At first sight, the apparation of the energies of both the initial and final states might be surprising. It is however needed in order to compensate the noninvariance of the volume, so that EV is a Lorentz invariant quantity. From now on we will always normalize our states to $2E$ (instead of 1 as is usually the case in nonrelativistic quantum mechanics).

We now substitute the definition (3.11) in Eq. (3.10) to get the fundamentally important expression,

$$\boxed{dw_{fi} = \frac{V^{1-n_i}}{(2\pi)^{3n_f-4}}\delta^{(4)}(p_f - p_i)\cdot|\mathcal{M}_{fi}|^2 \cdot \prod_{i=1}^{n_i}\frac{1}{2E_i}\prod_{f=1}^{n_f}\frac{d^3p_f}{2E_f}}. \tag{3.12}$$

We can then specify this result for the two cases of interest, as we do in the following subsections.

3.2.1 Total decay rate

In the case where $n_i = 1$, we view w_{fi} as a **decay rate** for the reaction,

$$a \to 1 + 2 + \cdots + n_f.$$

We have

$$\Gamma_{a\to\{n_f\}} = w_{\{f\}a} \qquad \text{(decay width)}, \tag{3.13}$$

$$\tau_{a\to\{n_f\}} = \frac{1}{\Gamma_{a\to\{n_f\}}} \qquad \text{(lifetime)}, \tag{3.14}$$

where $\{n_f\}$ stands for the n_f-particle final state $1 + 2 + \cdots + n_f$.

The next step is the definition of the **total decay width**,

$$\boxed{\Gamma_a = \sum_{\{n_f\}}\Gamma_{a\to\{n_f\}} = \frac{1}{2E_a}\frac{1}{(2\pi)^{3n_f-4}}\cdot\int\frac{d^3p_1}{2E_1}\cdots\frac{d^3p_{n_f}}{2E_{n_f}}\delta^{(4)}(p_f - p_i)|\mathcal{M}_{fi}|^2}, \tag{3.15}$$

and the lifetime

$$\tau_a = \frac{1}{\Gamma_a} \tag{3.16}$$

We remark that since E_a is not a Lorentz invariant quantity, Γ_a also depends on the reference frame. The quantity stated under the name "lifetime" in particle physics listings is always the lifetime as measured in the rest frame of the particle and is hence always the shortest one.

Example Without relativistic time dilation, one would expect the μ-leptons generated by cosmic rays in the high atmosphere and traveling almost at the speed of light to be able to travel $c\tau_\mu \approx 600\,\text{m}$ before decaying, making their detection on the earth surface almost impossible. When one takes time dilation into account, the distance becomes $c\tau_\mu \approx 10\,\text{km}$, which is in accordance with the observed μ-leptons number reaching the earth. This was actually for long the only available test of special relativity.

3.2.2 Scattering cross section

We now analyze the case of $n_i = 2$, i.e. the case of two particles interacting via the reaction,

$$a + b \to 1 + 2 + \cdots + n_f,$$

thus getting the **scattering cross section** $\sigma(a + b \to 1 + 2 + \cdots n_f)$ defined by,

$$\sigma = \frac{\#\text{ of transitions } a+b \to 1+2+\cdots n_f \text{ per unit time}}{\#\text{ of incoming particles per unit surface and time}} = \frac{w_{fi}}{\text{incoming flux}}. \tag{3.17}$$

The denominator can also be stated as,

$$\text{incoming flux} = (\text{number density}) \cdot (\text{relative velocity}) = \frac{v_{ab}}{V}.$$

Using Eqs. (2.12) and (3.17) we then find,

$$\boxed{\sigma_{i\to\{n_f\}} = \frac{1}{4F}\frac{1}{(2\pi)^{3n_f-4}} \int \left(\prod_{f=1}^{n_f} \frac{d^3 p_f}{2E_f}\right) \delta^{(4)}\left(\sum_{f=1}^{n_f} p_f - p_a - p_b\right) |\mathcal{M}_{fi}|^2}, \tag{3.18}$$

in which we see once more the Lorentz invariant Møller flux factor,

$$\begin{aligned} F &= E_a E_b v_{ab} = \sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2} \\ &= \sqrt{(s-(m_a+m_b)^2)(s-(m_a-m_b)^2)} \overset{s \gg m_a^2, m_b^2}{\longrightarrow} \frac{s}{2}. \end{aligned} \tag{3.19}$$

From the form of (3.18), we see that the total cross section is manifestly a Lorentz invariant quantity, since it only depends on Lorentz invariants.

3.2.3 Invariant phase space for n_f-particles

We have already seen that the scattering angle is related to the Mandelstam t-variable (Section 2.2.2).

In order to make the same statement for multiparticle final states, we define the n_f-particles phase space,

$$R_{n_f} = \int dR_{n_f} = \int \frac{d^3p_1}{2E_1} \cdots \frac{d^3p_{n_f}}{2E_{n_f}} \delta^{(4)} \left(\sum_{f=1}^{n_f} p_f - \sum_{i=1}^{n_i} p_i \right). \tag{3.20}$$

We now prove that R_n is a Lorentz invariant quantity.

$$\frac{d^3p_i}{2E_i} = \int_0^\infty dE_i \delta(p_i^2 - m_i^2) d^3p_i \tag{3.21}$$

$$= \int_{-\infty}^\infty \underbrace{d^4p_i}_{\text{L.I.}} \delta\underbrace{(p_i^2 - m_i^2)}_{\text{L.I.}} \underbrace{\theta(E_i)}_{E_i>0 \text{ is L.I.}} . \tag{3.22}$$

3.2.4 Differential cross section

In order to get the differential cross section, we define,

$$t_{jk} := (p_j - p_k)^2 = f(\angle(\vec{p}_j, \vec{p}_k)), \tag{3.23}$$

and write

$$\frac{d\sigma}{dt_{jk}} = \frac{1}{4F} \frac{1}{(2\pi)^{3n_f-4}} \int dR_{n_f} |\mathcal{M}_{fi}|^2 \delta(t_{jk} - (p_j - p_k)^2). \tag{3.24}$$

Starting from this expression, one can deduce differential distributions in all other kinematical variables (energies, angles) by expressing those through the t_{jk}'s.

3.3 $2 \to 2$ scattering cross section

Next we turn our attention towards the very important special case of $2 \to 2$ scattering, $n_i = n_f = 2$:

$$a + b \to 1 + 2.$$

3.3.1 Phase space

First, we take a look at the phase space R_2, we see that there are 6 integration variables and 4 constraints = 2 free parameters. The goal of the next steps will be to get rid of the

δ-functions.

$$
\begin{aligned}
R_2 &= \int \frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}\delta^{(4)}(p_1+p_2-p_a-p_b) \\
&\overset{(3.26)}{=} \int d^4p_1\delta(p_1^2-m_1^2)d^4p_2\delta(p_2^2-m_2^2)\theta(E_1)\theta(E_2)\delta^{(4)}(p_1+p_2-p_a-p_b) \\
&\overset{(3.27)}{=} \int d^4p_1\delta(p_1^2-m_1^2)\delta((p_a+p_b-p_1)^2-m_2^2)\theta(E_1)\theta(E_a+E_b-E_1) \\
&= \int_0^{E_a+E_b} dE_1 \int_0^{\infty} |\vec{p}_1|^2 d|\vec{p}_1| d\Omega \delta(E_1^2-\vec{p}_1^{\,2}-m_1^2)\delta((p_a+p_b-p_1)^2-m_2^2) \\
&\overset{(3.28)}{=} \int_0^{E_a+E_b} dE_1 d\Omega \underbrace{\frac{\sqrt{E_1^2-m_1^2}}{2}}_{\int_0^\infty |\vec{p}_1|^2\delta(E_1^2-\vec{p}_1^{\,2}-m_1^2)d|\vec{p}_1|} \delta\left(s-2\left((p_a+p_b)\cdot p_1+m_1^2m_2^2\right)\right)
\end{aligned} \tag{3.25}
$$

where we have used,

$$1 = \int \frac{dE_1}{2E_1}\delta(E_1^2-\vec{p}_1^{\,2}-m_1^2), \tag{3.26}$$

$$1 = \int d^4p_2\delta^{(4)}(p_1+p_2-p_a-p_b), \tag{3.27}$$

$$\delta(E_1^2-\vec{p}_1^{\,2}-m_1^2) = \frac{1}{|\vec{p}_1|}\left(\delta\left(|\vec{p}_1|-\sqrt{E_1^2-m_1^2}\right)+\underbrace{\delta\left(|\vec{p}_1|+\sqrt{E_1^2-m_1^2}\right)}_{=0,\text{ since } |\vec{p}_1|\geq 0}\right). \tag{3.28}$$

We did not make any assumption about the reference frame up to this point. We now specify our calculation for the center of mass frame,

$$\vec{p}_a+\vec{p}_b = 0 \Rightarrow E_a+E_b = \sqrt{s},$$

bringing Eq. (3.25) into,

$$
\begin{aligned}
R_2 &= \int_0^{\sqrt{s}} dE_1^* d\Omega^* \frac{|\vec{p}_1^{\,*}|}{2}\delta(s-2\sqrt{s}E_1^*+m_1^2-m_2^2) \\
&= \int d\Omega^* \frac{|\vec{p}_1^{\,*}|}{4\sqrt{s}} \\
&\Rightarrow dR_2 = \frac{1}{8s}\sqrt{\lambda(s,m_1^2,m_2^2)}d\Omega^*.
\end{aligned} \tag{3.29}
$$

For the last steps we used Eq. (2.10) and the fact that,

$$\delta(s-2\sqrt{s}E_1^*+m_1^2-m_2^2) = \frac{1}{2\sqrt{s}}\delta\left(E_1^*-\frac{1}{2\sqrt{s}}(s+m_1^2-m_2^2)\right).$$

A last step of the calculation can be made if the integrand has *no* angular dependency: since we are in the center of mass frame, we then have manifestly a 4π-symmetry and the scattering angle can take any value, the only restriction being that the two scattered particles are flying back-to-back in the center of mass frame. Therefore R_2 is then simply the integrand multiplied with the volume of the unit sphere, i.e.

$$R_2 = \int dR_2 = \frac{\pi}{2s}\sqrt{\lambda(s, m_1^2, m_2^2)}. \tag{3.30}$$

This simplification always applies for a $1 \to 2$ decay, but usually not for a $2 \to 2$ scattering reaction, where the incoming beam direction breaks the 4π-symmetry.

3.3.2 Differential cross section

Using Eq. (2.11) and (3.24) for $n_f = 2$, we get,

$$\frac{d\sigma}{d\Omega^*} = \frac{d\sigma}{dt}\frac{dt}{d\Omega^*} = \frac{|\vec{p}_1^{\,*}|}{64\pi^2 F\sqrt{s}}|\mathcal{M}_{fi}|^2, \tag{3.31}$$

resulting in the **differential cross section**,

$$\boxed{\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_1^{\,*}|}{|\vec{p}_a^{\,*}|}|\mathcal{M}_{fi}|^2}, \tag{3.32}$$

since from Eq. (2.12) $F = \sqrt{s}|\vec{p}_a^{\,*}|$.

For the special case of elastic scattering $|\vec{p}_1^{\,*}| = |\vec{p}_a^{\,*}|$, we get,

$$\boxed{\frac{d\sigma^{el.}}{d\Omega^*} = \frac{1}{64\pi^2 s}|\mathcal{M}_{fi}|^2}. \tag{3.33}$$

Finally, we write here the invariant differential cross section for future references,

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{16\pi s\sqrt{\lambda(s, m_1^2, m_2^2)}}|\mathcal{M}_{fi}|^2} \overset{s\gg m_1^2, m_2^2}{\longrightarrow} \frac{1}{16\pi s^2}|\mathcal{M}_{fi}|^2. \tag{3.34}$$

3.4 Unitarity of the $\mathcal{S}$-operator

We can compute the transition probability from the matrix elements for the transition $|i\rangle \to |f\rangle$,

$$|\mathcal{S}_{fi}|^2 = |\langle f|\,\mathcal{S}\,|i\rangle|^2, \tag{3.35}$$

$$\sum_f |\mathcal{S}_{fi}|^2 = 1, \tag{3.36}$$

where $\sum_f$ stands for

$$\sum_{\text{spins, particle types, quantum numbers}} \int \prod_f \left(\frac{V}{(2\pi)^3} d^3 p_f \right).$$

Developing and using the completeness relation,

$$\sum_f |f\rangle \langle f| = \mathbb{1},$$

we obtain

$$\begin{aligned} 1 &= \sum_f \langle i| \mathcal{S}^\dagger |f\rangle \langle f| \mathcal{S} |i\rangle = \langle i| \mathcal{S}^\dagger \mathcal{S} |i\rangle \qquad \forall |i\rangle \\ &\Rightarrow \boxed{\mathcal{S}^\dagger \mathcal{S} = \mathbb{1}}, \end{aligned} \tag{3.37}$$

in other words $\mathcal{S}$ is a unitary operator.

This important fact has profound implications. We state here two of them.

First, for two orthogonal states $|i\rangle$ and $|j\rangle$, we have,

$$\langle j| \mathcal{S}^\dagger \mathcal{S} |i\rangle = \langle j|i\rangle = \delta_{ij}.$$

The other implication concerns the expression introduced in Eq. (3.5),

$$\mathcal{S}_{fi} = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(p_f - p_i) \cdot \mathcal{T}_{fi}. \tag{3.38}$$

For a free theory, $\mathcal{T}_{fi} = 0$ and hence $\mathcal{S}_{fi} = \delta_{fi}$. On the other hand, for an interacting theory $\mathrm{Im}\, \mathcal{S}_{fi} \neq 0$.

An obvious comparison of the real and imaginary parts of $\mathcal{S}_{fi}$, tells us that,

$$\begin{aligned} \mathrm{Re}\, \mathcal{T}_{fi} &\rightsquigarrow \mathrm{Im}\, \mathcal{S}_{fi} \qquad &\text{(virtual contribution)}, \\ \mathrm{Im}\, \mathcal{T}_{fi} &\rightsquigarrow \mathrm{Re}\, \mathcal{S}_{fi} \qquad &\text{(absorbtive contribution)}. \end{aligned}$$

Taking a closer look at the absorbtive contribution, we get,

$$2i \mathrm{Im}\, \mathcal{T}_{fi} = \mathcal{T}_{fi} - \mathcal{T}^*_{fi} = i(2\pi)^4 \delta^{(4)}(p_f - p_i) \sum_n \mathcal{T}_{fn} \mathcal{T}^*_{in},$$

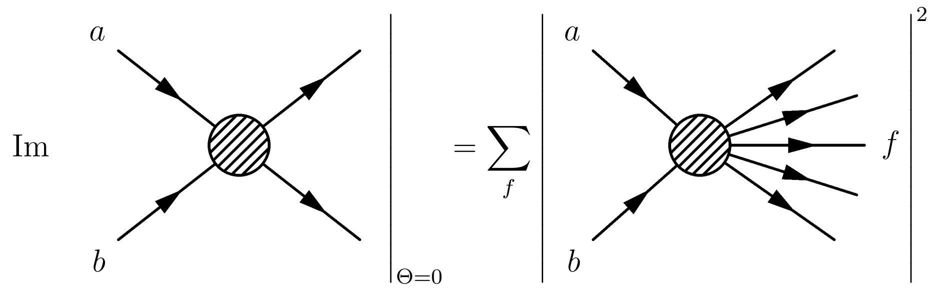
where n denotes an intermediate state.

The special case of elastic forward scattering ($|f\rangle = |i\rangle, \Theta^* = 0$) yields the surprising **optical theorem**,

$$\boxed{\mathrm{Im}\, \mathcal{M}_{ii} = \sqrt{\lambda(s, m_a^2, m_b^2)} \sigma_{tot}}, \tag{3.39}$$

relating a very specific element of $\mathcal{S}_{fi}$ with the total cross section for the transition $|i\rangle \to |f\rangle$, which is a measure for the probability for this transition to occur at all.

We can rewrite it symbolically with Feynman diagrams:



The computation of the matrix elements $\mathcal{M}_{fi}$ will be treated from Chapter 5 on.