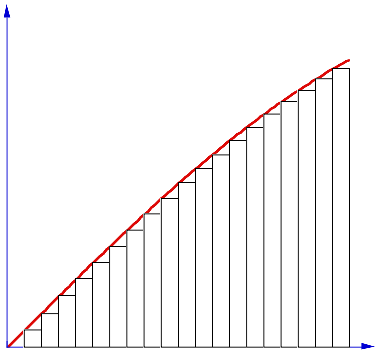
Numerical Integration

Numerical integration of a function

- Should be known from the numerics course in the first year
- is done by replacing the integral by a finite sum, such as:

$$\int_a^b f(x)dx = \frac{b-a}{N}\sum_{i=1}^{N} f\left(a + i\frac{b-a}{N}\right) + O(1/N)$$



Essential tricks for numerical integration

- Integrate as much as possible analytically

$$\int_0^1\int_0^1 y f(x)\,dx\,dy = \frac{1}{2}\int_0^1 f(x)\,dx$$

- Remove singularities

$$\int_{-1}^{1}|x| f(x)\,dx = \int_{-1}^{0}(\text{-x}) f(x)\,dx + \int_0^1 \text{x}\, f(x)\,dx = \int_0^1 \text{x}\,[f(x)+f(-x)]\,dx$$

$$\int_0^1 x^{1/3}\,dx = \int_0^1 3y^3\,dy$$

- Change of variables:
  - Stretch regions with large variations or large values
  - Shrink regions with small variations or small values

$$\int_0^\infty f(x)\exp(-x^2)\,dx = \int_0^1 f(-\ln(y))\exp(\ln(y)^2)/y\,dy$$

Higher order schemes

- Instead of rectangles

$$\int_a^b f(x)\,dx = \frac{b-a}{N}\sum_{i=1}^{N} f\left(a+i\frac{b-a}{N}\right) + O(1/N)$$

- Use
  - Trapezoidal rule

$$\int_a^b f(x)\,dx = \frac{b-a}{N}\left(\frac{1}{2}f(a) + \sum_{i=1}^{N-1} f\left(a+i\frac{b-a}{N}\right) + \frac{1}{2}f(b)\right) + O(1/N^2)$$

  - Or parabolas (Simpson rule)

$$\int_a^b f(x)\,dx = \frac{b-a}{3N}\left(f(a) + \sum_{i=1}^{N-1}(3-(-1)^i) f\left(a+i\frac{b-a}{N}\right) + f(b)\right) + O(1/N^4)$$

- for higher order schemes see the numerics course