Quantum Field Theory III

HS 10, Exercise sheet 13

Due date: 22.12.2010

Exercise 1: Supersymmetry and Strong-Electroweak Unification

Assume that the SM gauge Group $SU(3) \times SU(2)_L \times U(1)_Y$ is embedded in a simple group G, which is spontaneously broken at the GUT-scale M_X. The gauge couplings g_s of $SU(3)$, g of $SU(2)_L$ and g' of $U(1)_Y$ will be related by[1]

$$g_s^2 = g^2 = \frac{5}{3} g'^2 \quad \text{at Energies} \leq M_X.$$

At Energies far below M_X, these couplings will change, according to the one-loop renormalization group equations (RGEs)

$$\mu \frac{d}{d\mu} g'(\mu) = \beta_1 \left(g'(\mu)\right), \quad \mu \frac{d}{d\mu} g(\mu) = \beta_2 \left(g(\mu)\right), \quad \mu \frac{d}{d\mu} g_s(\mu) = \beta_3 \left(g_s(\mu)\right).$$

The beta functions are given by

$$\begin{aligned} \beta_1 &= \frac{g'^3}{4\pi^2} \left(\frac{5 n_g}{6} + \frac{n_s}{8} \right), \\ \beta_2 &= \frac{g^3}{4\pi^2} \left(-\frac{9}{6} + \frac{n_g}{2} + \frac{n_s}{8} \right), \\ \beta_3 &= \frac{g_s^3}{4\pi^2} \left(-\frac{9}{4} + \frac{n_g}{2} \right), \end{aligned} \tag{1}$$

where n_g is the number of generations of quarks and leptons and n_s is the number of Higgs doublets.

a) Solve the one-loop RGEs.

b) Find an expression for the unification scale M_X and the weak mixing angle θ by imposing the conditions

$$g(m_Z) = \frac{-e(m_Z)}{\sin(\theta)} \quad \text{and} \quad g'(m_Z) = \frac{-e(m_Z)}{\cos(\theta)}. \tag{2}$$

These expressions will still depend on n_g and on n_s. Imposing the MSSM conditions $n_g = 3$ and $n_s = 2$, calculate the numerical values for M_X and $\sin(\theta)$. How do these values agree with the experimental constraints coming for the proton decay and the electroweak precision tests?

[1] For an abelian gauge group the coupling is not quantized. The value 5/3 in front of g' is simply a convention.

Exercise 2: Superpotential Couplings in the MSSM
The most general MSSM superpotential respecting R-parity is

$$W_{MSSM} = \bar{u}\mathbf{Y}_u Q H_u - \bar{d}\mathbf{Y}_d Q H_d - \bar{e}\mathbf{Y}_e L H_d + \mu H_u H_d,$$

where the $\mathbf{Y}$s are 3×3 Yukawa matrices in family space. Approximating them by setting all but the $(3,3)$ entries to zero, we obtain

$$W_{MSSM} \sim y_t(\bar{t}tH_u^0 - \bar{t}bH_u^+) - y_b(\bar{b}tH_d^- - \bar{b}bH_d^0) - y_\tau(\bar{\tau}\nu_\tau H_d^- - \bar{\tau}\tau H_d^0) + \mu(H_u^+ H_d^- - H_u^0 H_d^0).$$

What interactions does this superpotential generate? What are the coupling constants for these interactions?