Quantum Field Theory III

HS 10, Exercise sheet 3

Due date: 13.10.2010

Exercise 1:
Write the supersymmetry algebra in 4-components spinor notation.

Hint: As done in exercise 3 on sheet 1 with the 2-components spinors, the fermionic generators Q^I_α and $\bar{Q}^{I\dot{\alpha}}$ have to be combined to a 4-components generator, i.e.

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad \Rightarrow \quad \mathcal{Q}^I \equiv \begin{pmatrix} Q^I_\alpha \\ \bar{Q}^{I\dot{\alpha}} \end{pmatrix}.$$

Similarly $\bar{\mathcal{Q}}^J \equiv \left(Q^{J\beta}, \bar{Q}^J_{\dot{\beta}}\right)$. Use the 2-components algebra

- $\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\right\} = 2\delta^{IJ}(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu,$
- $\left\{Q^I_\alpha, Q^J_\beta\right\} = \epsilon_{\alpha\beta}Z^{IJ},$
- $\left\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\right\} = \epsilon_{\dot{\alpha}\dot{\beta}}(Z^\star)^{IJ}$

to calculate $\left\{\mathcal{Q}^I, \bar{\mathcal{Q}}^J\right\}$?

Exercise 2:
Show that the massless supermultiplets with $N = 3$ and $N = 7$ have exactly the same particle content as the $N = 4$ and $N = 8$ supermultiplets respectively, when CPT invaricance is taken into account. For the $N = 3$ case you only need to consider global supersymmetry, i.e. $|\lambda| \leq 1$.

Exercise 3:
The raising and lowering operators of the massive multiplets in $N = 1$ supersymmetry obey the following commutation relations

$$\left\{a_\alpha, a^\dagger_\beta\right\} = \delta_{\alpha\beta}, \quad \{a_\alpha, a_\beta\} = \left\{a^\dagger_\alpha, a^\dagger_\beta\right\} = 0.$$

The Clifford vacuum is defined by

$$a_\alpha \left|\Omega\right\rangle = 0, \quad \alpha = 1, 2.$$

Show that such a state always exists in a representation.

Exercise 4: PhD Exercise[1]

In exercise 5 on sheet 2 it was shown that the square of the Pauli-Lubanski vector does not commute with the fermionic generators and therefore is not a casimir operator anymore. There is a generalization of the Pauli-Lubanski vector

$$\begin{aligned} C^2 &= C_{\mu\nu}C^{\mu\nu}, \\ C_{\mu\nu} &= B_\mu P_\nu - B_\nu P_\mu, \\ B_\mu &= W_\mu - \frac{1}{4}\bar{Q}_{\dot{\alpha}}\bar{\sigma}_\mu^{\dot{\alpha}\beta}Q_\beta, \end{aligned}$$

where $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}P^\nu M^{\rho\sigma}$ is the ordinary Pauli-Lubanski vector. Show that C^2 is a Casimir operator in supersymmetric theories. Moreover show that a particular state of a massive particle at rest is an eigenstate of the new Casimir operator with an eigenvalue that can be interpreted as a generalized spin, the "superspin".

[1] PhD Exercises are conceptually more interesting and usually more evolved than the ordinary exercises. They do not count as exercises Masters students have to solve in order to get their testat. The solution will be presented by a PhD student in the following exercise lesson.