Quantum Field Theory III

HS 10, Exercise sheet 4

Due date: 20.10.2010

Exercise 1:
The *little group* is the subgroup of the Lorentz group, which leaves the four-momentum invariant. Find the little group for

a) a massive particle,

b) a massless particle,

c) the vacuum.

Exercise 2:
The *Wess-Zumino Lagrangian* was introduced in the lecture

$$\mathcal{L}_{WZ} = \partial_\mu \phi^\dagger \partial^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + F^\dagger F.$$

Show that $\mathcal{L}_{WZ}$ is invariant under supersymmetry transformations up to a total derivative. Show that the following mass term is also invariant under supersymmetry transformations

$$\Delta\mathcal{L} = \left(m\phi F - \frac{1}{2} m\psi\psi \right) + \text{complex conjugate}.$$

What happens if you insert the equations of motion for F into $\Delta\mathcal{L}$?

Exercise 3: PhD Exercise on supersymmetric quantum mechanics
Look at a quantum mechanical system with commuting (x, y: bosonic) and anti-commuting (ξ, η: fermionic) coordinates

$$\mathcal{L} = \frac{1}{2}\dot{\mathrm{x}}^2 - \mathrm{V(x)} + \eta\dot{\xi} + \mathrm{W(x)}\eta\xi. \tag{1}$$

To quantize canonically, we postulate the (anti-) commutation relations,

$$[\hat{p}, \hat{x}] = i\hbar \qquad \{\hat{\eta}, \hat{\xi}\} = \hbar \qquad (\hat{\eta}^2 = \hat{\xi}^2 = 0). \tag{2}$$

The corresponding Hamiltonian is

$$H = \frac{1}{2}\hat{p}^2 + V(\hat{x}) - \frac{1}{2}W(\hat{x})[\hat{\eta}, \hat{\xi}]. \tag{3}$$

Look at the following representation for the fermionic Operators:

$$\hat{\xi} = \sigma_+ \text{ and } \hat{\eta} = \hbar\ \sigma_-. \tag{4}$$

The wave function is a 2-component spinor

$$\psi(x,t) = \begin{pmatrix} \phi_1(x,t) \\ \phi_2(x,t) \end{pmatrix}, \tag{5}$$

where ϕ_1 is the bosonic and ϕ_2 the fermionic component.

a) Express the Hamiltonian with the Pauli matrices

b) What is the relevance of the operator $N_F = \frac{\hat{\eta}\hat{\xi}}{2}$?

c) Look at the following potentials

$$V(\hat{x}) = \frac{1}{2}v^2(\hat{x}) \qquad \text{and} \qquad W(\hat{x}) = \frac{\partial v(\hat{x})}{\partial x} = v'(\hat{x})$$

and the operators

$$Q_i = \frac{1}{2}\sigma_i \left[\hat{p} + i\sigma_3 v(\hat{x})\right], \qquad \text{where} \quad i = 1,2 \tag{6}$$

Calculate $[H, Q_i]$ and $\{Q_i, Q_j\}$ and thus show that the Lagrangian is indeed supersymmetric.

d) Look at Q_i as matrices. How do they act on $\psi(x,t)$?

e) Look at the vacuum state $|0\rangle$, with $H|0\rangle = 0$. What can you deduce for $Q_i|0\rangle$?

f) Show that $|f\rangle = -\sqrt{\frac{2}{E}}Q_1|b\rangle = i\sqrt{\frac{2}{E}}Q_2|b\rangle$ is a fermionic state. What is the corresponding bosonic state?

g) What happens if the vacuum gets a positive energy?

h) What is the spectrum of supersymmetric quantum mechanics?