III. THE SUSY ALGEBRA

The Coleman-Mandula Theorem severely restricts the type of symmetries which can be present in a relativistic theory. The most general Lie algebra allowed is given by the Poincaré group and a group of internal symmetries whose generators B_a satisfy

$$\begin{cases} [B_a, B_b] = i f_{ab}{}^c B_c \\ [P_\mu, B_a] = 0 \\ [M_{\mu\nu}, B_a] = 0 \end{cases} .$$

However, by weakening one of the assumptions of the Coleman-Mandula theorem, we can find some more general symmetries compatible with a relativistic theory. This can be done by relaxing the assumption that the symmetry is described by a Lie group and by allowing some of the symmetry generators to be of "fermionic" nature, thus satisfying anticommutation relations instead of the usual commutation rules.

Haag, Lopuszanski and Sohnius (1975) proved that, with this weakened assumptions, supersymmetry is the only allowed extension of the usual Poincaré + internal symmetries algebra, moreover, the structure of the new symmetry is almost unique.

Before discussing the susy algebra, it is useful to introduce the definition of a graded Lie algebra, which will describe the structure of the susy generators.

- Let's start with the definition of a Lie Algebra:
  It is given by a vector space L (over a field like $\mathbb{R}$ or $\mathbb{C}$) with a composition rule

$$[\cdot,\cdot] : L \times L \to L$$

which satisfies the properties $(v_1, v_2, v_3 \in L)$

(i) $[v_1, v_2] \in L$

(ii) $[v_1, (v_2 + v_3)] = [v_1, v_2] + [v_1, v_3]$ (linearity)

(iii) $[v_1, v_2] = -[v_2, v_1]$ (antisymmetry)

(iv) $[v_1, [v_2, v_3]] + [v_3, [v_1, v_2]] + [v_2, [v_3, v_1]] = 0$ (Jacobi identity).

- Example. When we consider a matrix realization of a Lie Algebra, we identify the $[\cdot,\cdot]$ operation with the commutator. For example the space of complex 2×2 matrices which are traceless and anti-Hermitian form the Lie algebra of the group $SU(2,\mathbb{C})$, if we define

$$[a, b] \equiv ab - ba \quad , \quad a, b \in su(2,\mathbb{C}).$$

A basis for this algebra is given by $\tau_i = \frac{i}{2}\sigma_i$, where σ_i are the Pauli matrices.

· Now we can define a $\mathbb{Z}_2$-graded Lie algebra:

it is a vector space L which is the direct sum of two subspaces

$$L = L_0 \oplus L_1$$

with a composition law $[\cdot,\cdot\}$ satisfying

$$[L_0, L_0\} \subset L_0 \quad , \quad [L_0, L_1\} \subset L_1 \quad , \quad [L_1, L_1\} \subset L_0 .$$

The composition law $[\cdot,\cdot\}$ must satisfy also the properties

(i) $[L_i, L_j\} = -(-1)^{ij}[L_j, L_i\}$ (supersymmetrization)

(ii) $[L_i,[L_j,L_k\}\}(-1)^{ik} + [L_j,[L_k,L_i\}\}(-1)^{ji} + [L_k,[L_i,L_j\}\}(-1)^{kj} = 0$
(generalized Jacobi identities).

Notice that the subspace L_0, with the composition rule $[\cdot,\cdot\}$, forms an ordinary Lie algebra.

We can assign a degree to the elements of the algebra

· $\eta(x_a) = 0$ if $x_a \in L_0$ even or bosonic

· $\eta(x_a) = 1$ if $x_a \in L_1$ odd or fermionic

We can now define $[\cdot,\cdot\}$ as

$$[x_i, x_j\} = x_i x_j - (-1)^{\eta_i \eta_j} x_j x_i$$

where $\eta_i \equiv \eta(x_i)$, $\eta_j \equiv \eta(x_j)$. This composition rule satisfies the properties for a graded Lie algebra. More explicitly we get

commutators $\begin{cases} [x_i, x_j\} = x_i x_j - x_j x_i = [x_i, x_j] & x_i, x_j \in L_0 \\ [x_i, y_j\} = x_i y_j - y_j x_i = [x_i, y_j] & x_i \in L_0 ,\ y_j \in L_1 \end{cases}$

anticommutators $\big\{ \ [y_i, y_j\} = y_i y_j + y_j y_i = \{y_i, y_j\} \quad y_i, y_j \in L_1$

· The supersymmetry algebra

The susy algebra is a Z_2-graded Lie algebra with

· a bosonic sector given by
- Poincaré P_μ, $M_{\mu\nu}$
- Internal symmetries B

· an extra fermionic sector with generators Q^I_α, $\bar{Q}^I_{\dot\alpha}$, $I = 1, \dots, N$.

· We must now derive the commutation/anticommutation relations among the generators.

• We start from the anticommutator

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\}.$$

If we evaluate the anticommutator on a state of the Hilbert space of physical states we get, for the $I=J$ case,

$$\langle u| \{Q^I_\alpha, \bar{Q}^I_{\dot\alpha}\} |u\rangle = |\bar{Q}^I_{\dot\alpha}|u\rangle|^2 + |Q^I_\alpha|u\rangle|^2 > 0 \quad \text{if} \quad Q^I_\alpha \neq 0,$$

where we used the fact that $\bar{Q} = Q^+$.

This means that $\{Q^I_\alpha, \bar{Q}^I_{\dot\alpha}\}$ can not vanish if $Q^I_\alpha \neq 0$.

Now we assume that Q^I is in the (j, j') representation of the Lorentz group, in this case $\bar{Q}^I$ is in the (j', j) representation and $\{Q^I, \bar{Q}^I\}$ will contain the representation $(j+j', j+j')$. This is a bosonic representation, so it is part of the bosonic subalgebra which satisfies the Coleman-Mandula theorem. The only bosonic generator of this kind is P_μ, which is in the $(1/2, 1/2)$ representation, thus all the Q's must be in the $(1/2, 0)$ or $(0, 1/2)$ representation.

By imposing Lorentz invariance we get

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} = 2\,\delta^{IJ} (\sigma^\mu)_{\alpha\dot\alpha} P_\mu,$$

where we used the freedom in the choice of the generators to fix the normalization and to get a δ^{IJ} matrix (see for example Sohnius).

- The commutators of Q^I and $\bar{Q}^I$ with $M_{\mu\nu}$ are determined by the fact that

$$Q^I \sim (1/2, 0) \text{ representation},$$
$$\bar{Q}^I \sim (0, 1/2) \text{ representation}.$$

To write the commutators we need to rewrite the Lorentz generators by using the 2-component notation.

The usual Lorentz generators are

$$\Sigma^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu}$$

where

$$\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{1}{2}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}.$$

we see that dotted and undotted spinors transform separately:

- ψ_α has generators $i\sigma^{\mu\nu}$

$$(\sigma^{\mu\nu})_\alpha{}^\beta \equiv \frac{1}{4}\left(\sigma^\mu_{\alpha\dot\gamma}\,\bar{\sigma}^{\nu\dot\gamma\beta} - \sigma^\nu_{\alpha\dot\gamma}\,\bar{\sigma}^{\mu\dot\gamma\beta}\right),$$

- $\bar{\psi}^{\dot\alpha}$ has generators $i\bar{\sigma}^{\mu\nu}$

$$(\bar{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \equiv \frac{1}{4}\left(\bar{\sigma}^{\mu\dot\alpha\gamma}\,\sigma^\nu{}_{\gamma\dot\beta} - \bar{\sigma}^{\nu\dot\alpha\gamma}\,\sigma^\mu{}_{\gamma\dot\beta}\right).$$

The commutators of Q and $\bar{Q}$ with $M_{\mu\nu}$ are given by

$$\begin{cases} \underline{[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta} \\ \underline{[M_{\mu\nu}, \bar{Q}^{I\dot\alpha}] = i(\bar{\sigma}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,\bar{Q}^{I\dot\beta}} \end{cases}$$

- To complete the transformation properties of the Q's under the Poincaré group, we need to find the commutation rules with P_μ. We will prove that

$$\begin{cases} \underline{[Q^I_\alpha, P_\mu] = 0}, \\ \underline{[\bar{Q}^{I\dot\alpha}, P_\mu] = 0}. \end{cases}$$

<u>Proof.</u> The commutator of Q with P_μ can contain the representations $(1/2, 1/2)\otimes(1/2, 0) = (1, 1/2)\oplus(0, 1/2)$.
There are no $(1, 1/2)$ generators, so we get

$$[Q^I_\alpha, P_\mu] = C^I{}_J(\sigma_\mu)_{\alpha\dot\beta}\,\bar{Q}^{J\dot\beta},$$

and its adjoint

$$[\bar{Q}^{I\dot\alpha}, P_\mu] = (C^I{}_J)^*(\bar{\sigma}_\mu)^{\dot\alpha\beta}\,Q^J_\beta.$$

We have

$$[[Q^I_\alpha, P_\mu], P_\nu] = C^I{}_J\,(C^J{}_K)^*(\sigma_\mu)_{\alpha\dot\beta}(\bar{\sigma}_\nu)^{\dot\beta\gamma}\,Q^K_\gamma.$$

By using the Jacobi identity

$$[[Q^I_\alpha, P_\mu], P_\nu] + [[P_\mu, P_\nu], Q^I_\alpha] + [[P_\nu, Q^I_\alpha], P_\mu] = 0$$

we get

$$c^I{}_J (c^J{}_K)^* (\sigma_\mu \bar\sigma_\nu)_\alpha{}^\gamma Q^K{}_\gamma - c^I{}_J (c^J{}_K)^* (\sigma_\nu \bar\sigma_\mu)_\alpha{}^\gamma Q^K{}_\gamma =$$

$$= C^I{}_J (C^J{}_K)^* \left(\sigma_\mu \bar\sigma_\nu - \sigma_\nu \bar\sigma_\mu\right)_\alpha{}^\gamma Q^K{}_\gamma$$

$$= 4\, (CC^*)^I{}_K\, (\sigma_{\mu\nu})_\alpha{}^\gamma Q^K{}_\gamma = 0$$

Given that $\sigma_{\mu\nu}$ and Q are non-zero, we have

$$CC^* = 0. \qquad (*)$$

Let's now consider the most general form of $\{Q^I_\alpha, Q^J_\beta\}$:

$$\{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta} Z^{IJ} + \text{term symmetric in } \alpha\beta,$$

where $Z^{IJ} = -Z^{JI}$ is in the $(0,0)$ representation. Then

$$0 = \varepsilon^{\alpha\beta} [\{Q^I_\alpha, Q^J_\beta\}, P_\mu] = \varepsilon^{\alpha\beta}\left(\{Q^I_\alpha, [Q^J_\beta, P_\mu]\} + \{Q^J_\beta, [Q^I_\alpha, P_\mu]\}\right)$$

$$= \varepsilon^{\alpha\beta}\left(c^J{}_K (\sigma_\mu)_{\beta\dot\gamma} \{Q^I_\alpha, \bar Q^{K\dot\gamma}\} + c^I{}_K (\sigma_\mu)_{\alpha\dot\gamma} \{Q^J_\beta, \bar Q^{K\dot\gamma}\}\right)$$

$$= \varepsilon^{\alpha\beta}\left(C^J{}_K (\sigma_\mu)_{\beta\dot\gamma} \varepsilon^{\dot\gamma\dot\delta}\, 2\, \delta^{IK} (\sigma^\rho)_{\alpha\dot\delta} P_\rho + C^I{}_K (\sigma_\mu)_{\alpha\dot\gamma} \varepsilon^{\dot\gamma\dot\delta}\, 2\, \delta^{JK} (\sigma^\rho)_{\beta\dot\delta} P_\rho\right)$$

$$= \left(C^J{}_I - C^I{}_J\right) (\sigma_\mu)_{\beta\dot\gamma} (\sigma^\rho)_{\alpha\dot\delta}\, \varepsilon^{\alpha\beta} \varepsilon^{\dot\gamma\dot\delta} P_\rho,$$

which follows from the fact that Z^{IJ} is a combination of the bosonic internal generators, which commute with P_μ. From the above relation

$$C^I{}_J = C^J{}_I,$$

and hence (from $(*)$) we get

$$CC^+ = 0 \quad \Rightarrow \quad \underline{C = 0}$$

This completes the proof. ▨

· Let's now consider the $\{Q^I_\alpha, Q^J_\beta\}$ anticommutator. It must be a linear combination of the representations

$$(\tfrac{1}{2},0)\otimes(\tfrac{1}{2},0) = (0,0)+(1,0).$$

The only $(1,0)$ generator is $M_{\mu\nu}$, but it does not commute with P_μ, while

$$[\{Q^I_\alpha, Q^J_\beta\}, P_\mu] = 0$$

which follows from $[Q^I_\alpha, P_\mu]=0$. Thus we are left with

$$\underline{\{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta}\, Z^{IJ}}$$

with Z^{IJ} a linear combination of the internal symmetry generators

$$Z^{IJ} = a^{r\,IJ}\, B_r\,,$$

which satisfy $Z^{IJ} = -Z^{JI}$. As we will prove in the following

$$\underline{[Z^{IJ}, \text{any generator}] = 0\,,}$$

which means that the Z^{IJ} form an Abelian subalgebra. For this reason the Z^{IJ} are called <u>central charges</u>.

· The central charges commute with the momentum generators P_μ because of the Coleman-Mandula theorem. Z^{IJ} are bosonic generators in the $(0,0)$ representation of the Lorentz group, thus they are a linear combination of internal generators and

$$\underline{[P_\mu, Z^{IJ}] = [P_\mu, (Z^{IJ})^*] = 0}$$

· By using the Jacobi identity for Q^I_α, Q^J_β and $\bar{Q}^K_{\dot\gamma}$ we easily get (<u>exercise</u>)

$$\underline{[\bar{Q}^K_{\dot\gamma}, Z^{IJ}] = 0}$$

and analogously

$$\underline{[Q^K_\gamma, (Z^{IJ})^*] = 0\,.}$$

· Let's now consider the Jacobi identity for Z^{IJ}, Q^K_α and $\bar{Q}^L_{\dot\beta}$:

$$-[Z^{IJ}, \cancel{\{Q^K_\alpha, \bar{Q}^L_{\dot\beta}\}}] + \{\bar{Q}^L_{\dot\beta}, [Z^{IJ}, Q^K_\alpha]\} - \{Q^K_\alpha, \cancel{[\bar{Q}^L_{\dot\beta}, Z^{IJ}]}\} = 0\,,$$

which implies

$$\{\bar{Q}^L_{\dot\beta}, [Z^{IJ}, Q^K_\alpha]\} = 0\,. \qquad (*)$$

The most general form of the commutator between Z^{IJ} and Q^K_α is

$$[Z^{IJ}, Q^K_\alpha] = \sum_L M^{IJKL}\, Q^L_\alpha\,,$$

and substituting it in $(*)$ we get

$$0 = \{\bar{Q}^L_{\dot\beta}, \sum_N M^{IJKN} Q^N_\alpha\} = \sum_N M^{IJKN}\, 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{LN} = 2M^{IJKL}\sigma^\mu_{\alpha\dot\beta} P_\mu.$$

This relation implies that $M^{IJKL}=0$, hence

$$\underline{[Z^{IJ}, Q^K_\alpha] = 0}.$$

Analogously one can prove that $\underline{[(Z^{IJ})^*, \bar{Q}^K_{\dot\alpha}]=0}$.

· By using the Jacobi identity among Z^{IJ}, Q^K_α and Q^L_β we obtain (<u>exercise</u>)

$$\underline{[Z^{IJ}, Z^{KL}] = 0},$$

and analogously for the complex conjugate relation $\underline{[(Z^{IJ})^*, (Z^{KL})^*]=0}$.

· Moreover from the Jacobi identity involving Z^{IJ}, $\bar{Q}^K_{\dot\alpha}$, $\bar{Q}^L_{\dot\beta}$ we get (<u>exercise</u>)

$$\underline{[Z^{IJ}, (Z^{KL})^*] = 0}.$$

· Finally we must find the commutation relations of the supersymmetry generators with the generators of the internal symmetries.

Given that the internal symmetries generators are in the $(0,0)$ representation we can write

$$\begin{cases} [Q^I_\alpha, B_\ell] = S_\ell{}^I{}_J\, Q^J_\alpha \\ [\bar{Q}^I_{\dot\alpha}, B_\ell] = -\bar{Q}^J_{\dot\alpha}(S_\ell{}^I{}_J)^* \end{cases}$$

By considering the Jacobi identity with B_ℓ, Q^I_α and $\bar{Q}^J_{\dot\beta}$ we get

$$0 = [B_\ell, \{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\}] + \{Q^I_\alpha, [\bar{Q}^J_{\dot\beta}, B_\ell]\} - \{\bar{Q}^J_{\dot\beta}, [B_\ell, Q^I_\alpha]\}$$

$$= 2\sigma^\mu_{\alpha\dot\beta}\delta^{IJ}\cancel{[B_\ell, P_\mu]} + \{Q^I_\alpha, -\bar{Q}^K_{\dot\beta}(S_\ell{}^J{}_K)^*\} + \{\bar{Q}^J_{\dot\beta}, S_\ell{}^I{}_K Q^K_\alpha\}$$

$$= -(S_\ell{}^J{}_K)^* \cdot 2\sigma^\mu_{\alpha\dot\beta}\delta^{IK} P_\mu + S_\ell{}^I{}_K\, 2\sigma^\mu_{\alpha\dot\beta}\delta^{JK} P_\mu$$

$$\Rightarrow \quad \left(S_\ell{}^I{}_J - (S_\ell{}^J{}_I)^*\right)\cdot 2\sigma^\mu_{\alpha\dot\beta} P_\mu = 0$$

From this we get

$$S_\ell{}^I{}_J = (S_\ell{}^J{}_I)^* = S^+_\ell{}_I{}^J \quad \Rightarrow \quad \underline{S_\ell = S_\ell^+}$$

This implies the relations:

$$\begin{cases} \underline{[Q^I_\alpha, B_\ell] = S_\ell{}^I{}_J\, Q^J_\alpha} \\ \underline{[\bar{Q}^I_{\dot\alpha}, B_\ell] = -\bar{Q}^J_{\dot\alpha}\, S_\ell{}_J{}^I} \end{cases}$$

- From the Jacobi identity with B_a, B_b and Q^I_α, one can prove that (exercise)

$$\underline{[S_a, S_b] = i f_{ab}{}^c S_c},$$

where $f_{ab}{}^c$ are the structure constants of the internal symmetry group

$$[B_a, B_b] = i f_{ab}{}^c B_c .$$

This means that the S_a form a representation of the internal symmetry algebra.

- Now we can also prove that the central charges Z^{IJ} commute also with the internal symmetry generators. We start from the Jacobi identity

$$0 = \{Q^I_\alpha, [Q^J_\beta, B_\ell]\} - \{Q^J_\beta, [B_\ell, Q^I_\alpha]\} + [B_\ell, \{Q^I_\alpha, Q^J_\beta\}]$$

$$= S_\ell{}^J{}_K \{Q^I_\alpha, Q^K_\beta\} + S_\ell{}^I{}_K \{Q^J_\beta, Q^K_\alpha\} + [B_\ell, \epsilon_{\alpha\beta} Z^{IJ}]$$

$$= \epsilon_{\alpha\beta}\left([B_\ell, Z^{IJ}] + S_\ell{}^J{}_K Z^{IK} - S_\ell{}^I{}_K Z^{JK}\right)$$

$$\Rightarrow \underline{[B_\ell, Z^{IJ}] = S_\ell{}^J{}_K Z^{KI} - S_\ell{}^I{}_K Z^{KJ}} .$$

Together with the previous result $[Z^{IJ}, Z^{KL}] = 0$, we get that the Z^{IJ} form an <u>invariant Abelian subalgebra</u> of the bosonic symmetry algebra.

The Coleman-Mandula theorem states also that the complete bosonic internal symmetries are given by the direct product of a semi-simple Lie algebra and several $U(1)$ algebras. The only invariant Abelian subalgebras of such a Lie algebra are spanned by $U(1)$ generators, so the Z^{IJ} must be $U(1)$ generators and must commute with the B_ℓ:

$$\underline{[B_\ell, Z^{IJ}] = 0} .$$

<u>NOTE.</u> Although the central charges commute with all the generators, they are not just numbers. They are operators which can take different values on different physical states.
Notice that on a supersymmetric vacuum, the central charges vanish:

$$\epsilon_{\alpha\beta} Z^{IJ} |0\rangle = \{Q^I_\alpha, Q^J_\beta\}|0\rangle = 0$$

because $Q^K_\gamma |0\rangle = 0$ for a supersymmetric vacuum.

- *R-symmetry*

In the absence of central charges the supersymmetry algebra is invariant under a group $U(N)$ of internal symmetries, where N is the number of supersymmetry generators Q^I_α:

$$Q^I_\alpha \to \sum_J V^I{}_J Q^J_\alpha$$

with $V^I{}_J$ an $N\times N$ unitary matrix.

A supersymmetry algebra with $N>1$ is called an *N-extended supersymmetry*, while for $N=1$ we have a *simple supersymmetry*.

- If we have just one fermionic generator Q_α ($N=1$), we can not have central charges as a consequence of the relation

$$Z^{IJ} = - Z^{JI}.$$

In this case we have a simplified form of the supersymmetry algebra

$$\begin{cases} \{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \\ \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \end{cases}$$

In this case the R-symmetry is just a $U(1)$ symmetry

$$Q_\alpha \to \exp(i\varphi)\, Q_\alpha$$

with φ a real phase.

· Properties of the supersymmetry algebra

- Positivity of the energy

In a supersymmetric theory the energy P_0 is always positive. To see this we consider any state $|\Phi\rangle$, then, by the positivity of the Hilbert space we have

$$0 \leq \| Q^I_\alpha |\Phi\rangle\|^2 + \| (Q^I_\alpha)^+ |\Phi\rangle \|^2$$

$$= \langle \Phi | \left((Q^I_\alpha)^+ Q^I_\alpha + Q^I_\alpha (Q^I_\alpha)^+ |\Phi\rangle \right)$$

$$= \langle \Phi | \{ Q^I_\alpha , \bar{Q}^I_{\dot\alpha} \} | \Phi \rangle$$

$$= 2 \sigma^\mu_{\alpha\dot\alpha} \langle \Phi | P_\mu | \Phi \rangle ,$$

where we used the fact that

$$(Q^I_\alpha)^+ = \bar{Q}^I_{\dot\alpha} .$$

Taking the trace over the α and $\dot\alpha$ indices, and using $\mathrm{tr}\, \sigma^\mu = 2\delta^{\mu 0}$, we get

$$\underline{0 \leq 4 \langle \Phi | P_0 | \Phi \rangle} ,$$

which proves the positivity of the energy on any physical state.

· Notice that the relation

$$\{ Q^I_\alpha , \bar{Q}^J_{\dot\alpha} \} = 2 \sigma^\mu_{\alpha\dot\alpha} P_\mu$$

has another important consequence: If the vacuum state of the theory $|0\rangle$ is supersymmetric (that is, if supersymmetry is <u>not spontaneously broken</u>) then

$$Q^I_\alpha |0\rangle = \bar{Q}^I_{\dot\alpha} |0\rangle = 0$$

thus

$$\underline{\langle 0 | P_0 | 0 \rangle = 0} ,$$

hence <u>the vacuum has zero energy</u>.

- Casimirs of the supersymmetry algebra.

It is easy to prove that P^2 is still a Casimir of the supersymmetry algebra (<u>exercise</u>).

On the other hand W^2 does not commute with the supersymmetry generators, so it is <u>not</u> a Casimir of the supersymmetry algebra (<u>exercise</u>):

$$[W^2, Q] \neq 0 .$$

This implies that all the states in a representation will have the same mass, but <u>not</u> the same spin.