IV. REPRESENTATIONS OF THE SUSY ALGEBRA

· The "Fermions" = "Bosons" rule

Before discussing the SUSY representations it is useful to derive an important property of susy supermultiplets:

· A supermultiplet always contains an equal number of bosonic and fermionic degrees of freedom.

Proof. Let the fermion number be N_F equal to 1 on a fermionic state and 0 on a bosonic state, so that $(-1)^{N_F}$ is +1 for a boson and −1 for a fermion. We need to show that

$$\mathrm{Tr}\,(-1)^{N_F}=0$$

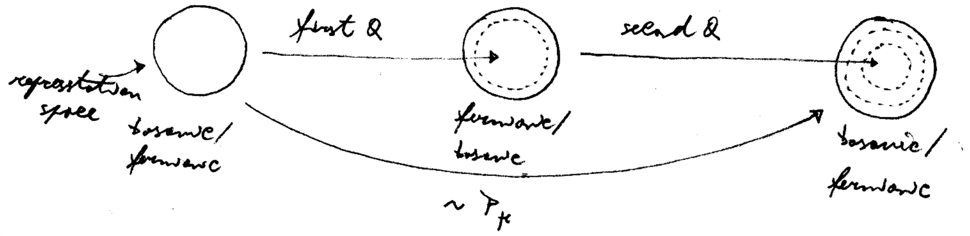
on a finite dimensional representation of the susy algebra. From the fact that $(-1)^{N_F}$ anticommutes with Q, and using the cyclicity of the trace, one has

$$0=\mathrm{Tr}\left(-Q_\alpha(-)^{N_F}\bar{Q}_{\dot\beta}+(-)^{N_F}\bar{Q}_{\dot\beta}Q_\alpha\right)=\mathrm{Tr}\left((-)^{N_F}\{Q_\alpha,\bar{Q}_{\dot\beta}\}\right)$$

$$=2\sigma^\mu_{\alpha\dot\beta}\,\mathrm{Tr}\left((-)^{N_F}P_\mu\right).$$

In any representation in which P_μ is non-zero, we get the wanted result. ▨

We can also understand the "Fermions" = "Bosons" rule pictorially. The anticommutators $\{Q,\bar{Q}\}$ are a combination of two mappings



But the anticommutator $\{Q_\alpha,\bar{Q}_{\dot\beta}\}=2\sigma^\mu_{\alpha\dot\beta}P_\mu$ means that if P_μ maps the representation space onto itself, this implies that the bosonic and fermionic subspaces must have the same dimension and Q must map one onto the other.

For a large class of representations P_μ gives a map onto the representation space itself, so the "Fermions" = "Bosons" rule is verified. In particular this is true on quantum fields on which the momentum P_μ is the generator of translations and is represented by the derivatives $P_\mu\sim i\partial_\mu$.

· In the following we will discuss the irreducible representations of the susy algebra on single particle states. This means that we will consider representations on asymptotic on-shell physical states.

Susy representations on quantum fields will be discussed afterwards.

· Massless supermultiplets.

We will start by considering the representations for massless supermultiplets. $P^2 = m^2$ is a Casimir operator for the susy algebra, hence all the particles in a supermultiplet will have the same mass.

For massless states we can choose a reference frame with

$$\underline{P_\mu = (E, 0, 0, E)}$$

which, of course, satisfies $P_\mu P^\mu = 0$. Then we get

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}_{\alpha\beta} \delta^{IJ} .$$

In particular we have that

$$\{Q^I_1, \bar{Q}^J_{\dot 1}\} = 0 \qquad \forall I, J.$$

This has an important consequence: on a positive definite Hilbert space we must set

$$Q^I_1 = \bar{Q}^J_{\dot 1} = 0 \qquad \forall I, J,$$

as can be seen from

$$0 = \langle \Phi | \{Q^I_1, \bar{Q}^J_{\dot 1}\} | \Phi \rangle = \| Q^I_1 |\Phi\rangle \|^2 + \| \bar{Q}^J_{\dot 1} |\Phi\rangle \|^2 \Rightarrow Q^I_1 = \bar{Q}^J_{\dot 1} = 0 .$$

Thus we are left with only N fermionic generators:

$$\underline{Q^I_2 \text{ and } \bar{Q}^J_{\dot 2}} .$$

We can rescale them as

$$a_I \equiv \frac{1}{\sqrt{4E}} Q^I_2 \quad , \quad a^+_I \equiv \frac{1}{\sqrt{4E}} \bar{Q}^I_{\dot 2} ,$$

in this case a_I and a^+_I are anticommuting annihilation and creation operators

$$\{a_I, a^+_J\} = \delta_{IJ} \quad , \quad \{a_I, a_J\} = \{a^+_I, a^+_J\} = 0 .$$

We can construct a supermultiplet by acting with the Q^I_2 and $\bar{Q}^J_{\dot 2}$ on one of its states. Given that Q^I and $\bar{Q}^I$ commute with P_μ, all the states in a multiplet will have the same P_μ.

The building blocks to construct the supermultiplets are the massless representations of the Poincaré group, which are characterized by $P^2 = 0$ and by some helicity λ.

The commutation relations of the helicity operator, which in the frame we chose is $J_3 = M_{12}$, with the Q^I_2 and $\bar{Q}^J_{\dot 2}$

$$[M_{12}, Q^I_2] = -\frac{1}{2} Q^I_2$$

$$[M_{12}, \bar{Q}^J_{\dot 2}] = \frac{1}{2} \bar{Q}^J_{\dot 2}$$

tell us that Q^I_2 lowers the helicity by 1/2 and $\bar{Q}^J_{\dot 2}$ raises it by 1/2 ;

so that

$$Q^I_2 |\lambda\rangle = |\lambda - 1/2\rangle$$
$$Q^{\bar{I}}_2 |\lambda\rangle = |\lambda + 1/2\rangle .$$

To construct the supermultiplet we will start from the state with lowest helicity $|\lambda_0\rangle$, which is annihilated by all the a_I (this is the "Clifford vacuum"):

$$\underline{a_I |\lambda_0\rangle = 0} .$$

We assume that $|\lambda_0\rangle$ is a singlet of the $SU(N)$ symmetry which acts on the I and J indices. The other states in the supermultiplet can be obtained by acting with a_I^+ on $|\lambda_0\rangle$:

$$|\lambda_0\rangle$$
$$a_I^+ |\lambda_0\rangle = |\lambda_0 + 1/2\rangle_I$$
$$a_I^+ a_J^+ |\lambda_0\rangle = |\lambda_0 + 1\rangle_{IJ}$$
$$\vdots$$
$$a_1^+ a_2^+ \dots a_N^+ |\lambda_0\rangle = |\lambda_0 + N/2\rangle .$$

Due to the antisymmetry in $I, J, \dots$ there are $\binom{N}{k}$ states with helicity $\lambda = \lambda_0 + k/2$, $k = 0, 1, \dots, N$. In total a supermultiplet contains 2^N states:

$$\left.\begin{matrix} 2^{N-1} \text{ bosons} \\ 2^{N-1} \text{ fermions} \end{matrix}\right\} \; 2^N \text{ states.}$$

In general such a supermultiplet, except if $\lambda_0 = -N/4$, the helicities will not be distributed symmetrically around zero. Such supermultiplets can not be invariant under CPT, since CPT flips the sign of the helicity. To satisfy CPT we then need to double these multiplets by adding their CPT conjugate with opposite helicities and opposite quantum numbers.

. <u>simple supersymmetry : $N=1$ case</u>

For simple supersymmetry each massless supermultiplet contains two states

$$|\lambda_0\rangle \quad , \quad |\lambda_0 + 1/2\rangle .$$

They can never be CPT self-conjugate, so we need to double them. We have the following supermultiplets:

- <u>Chiral multiplet</u>: $\lambda_0 = 0$, so we have $(0, 1/2) \oplus (-1/2, 0)$, or, in other words a Weyl fermion and a complex scalar;

- <u>Vector multiplet</u>: $\lambda = 1/2$, we have $(1/2, 1) \oplus (-1, -1/2)$, that is a massless vector and a Weyl fermion;

- <u>Gravitino multiplet</u>: $\lambda = 1$, so that $(1, 3/2) \oplus (-3/2, -1)$, i.e. a gravitino and a massless vector;

- <u>Graviton multiplet</u>: $\lambda = 3/2$, contains $(3/2, 2) \oplus (2, -3/2)$, corresponding to the graviton and the gravitino.

Massless particles with spin greater than 2 can not be consistently included in an interacting theory. Thus the only allowed massless supermultiplets are the ones listed before.

In a renormalizable theory without gravity we can have only chiral and vector multiplets. If we also consider gravity (getting a so called supergravity theory) we also need the supermultiplets with higher helicity.

- Extended supersymmetry : $N>1$ case.

Because the Q^I_2 and $\bar{Q}^I_{\dot 2}$ all give zero on the states of a supermultiplet (including the states obtained by acting with Q^I_2, $\bar{Q}^I_{\dot 2}$ on any state of the multiplet), the central charges Z^{IJ} must also give zero on any state of the multiplet. This is the reason for which we did not include them in the previous discussion.

The algebra of the N raising operators a^+_I is invariant under an $SU(N)$ R-symmetry. This implies that the states of given helicity in a supermultiplet form a representation of $SU(N)$, namely the rank n antisymmetric tensor representation (given that the a^+_I anticommute).

Now we briefly discuss the most relevant supermultiplets in the $N=2$, $N=4$ and $N=8$ cases.

- $N=2$

There are two multiplets for global supersymmetry:

- vector multiplet, which contains:
  - a gauge boson (massless vector), helicity $+1$;
  - two fermions of helicity $+1/2$, which form a doublet of the $SU(2)$ R-symmetry;
  - one boson of helicity 0;

  to get a CPT-invariant multiplet we must also add the conjugate multiplet with reversed helicities.

- hypermultiplet, which contains
  - one fermion of each helicity $\pm 1/2$;
  - an $SU(2)$ doublet of bosons with helicity 0.

  To have a CPT-invariant multiplet in a quantum field theory, we must add the conjugate multiplet. This is because otherwise the scalars would be just two real scalar fields which can not form an $SU(2)$ doublet.

- $N=4$

If we are interested in global supersymmetry, we can use only one $N=4$ supermultiplet:

- it is CPT self-conjugate and contains
  - a massless vector $(-1,+1)$;
  - 4 fermions of each helicity $(-1/2, +1/2)$;
  - 6 bosons of helicity 0.

- $N=8$

In this case there is only one multiplet with helicities $|\lambda| \leq 2$. It is CPT self-conjugate and contains:

- 1 graviton with helicities $(-2,+2)$;
- 8 gravitinos with helicities $(-3/2, +3/2)$;
- 28 gauge bosons with helicities $(-1,+1)$;
- 56 fermions with helicities $(-1/2, +1/2)$;
- 70 bosons with helicity 0.

This means that in the $N=8$ case we can only have supergravity theories and we can not build a theory with only global supersymmetry.

- NOTE. The supermultiplets in the $N=3$ and $N=7$ extended supersymmetries, when CPT invariance is taken into account, have exactly the same particle content as the $N=4$ and $N=8$ supermultiplets respectively. (exercise)

- Chiral fermions

In simple susy we can have chiral fermions by using the supermultiplets containing just helicity $+1/2$ and 0. These can be in a complex representation of the gauge group, distinct from the representation of the CPT-conjugate supermultiplet.

In all the supermultiplets of extended susy (except for hypermultiplets of $N=2$) fermions of spin $1/2$ are always in multiplets which contain gauge bosons. This means that they are in the adjoint representation of the gauge group, which is a real representation, and they can not be chiral (which would require them to be in a complex representation).

In the Standard Model fermions are in chiral representations of the $SU(2) \times U(1)$ gauge group, so extended susy is in conflict with the chiral nature of quarks and leptons.

Also the hypermultiplet of $N=2$ susy can not give chiral fermions. Each multiplet contains fermions of helicity $+1/2$ and $-1/2$, therefore they must transform in the same way under gauge transformations that leave the supersymmetry generators invariant. They may belong to a complex representation, but then the CPT conjugate of this hypermultiplet would be in the complex-conjugate representation, and in that case the sum of the two representation would give again a real representation without chiral fermions.

- Massive supermultiplets

The building blocks to construct massive susy representations are the usual massive Lorentz group representations, which are characterized by a certain mass $P^2 = m^2$ and a given spin J.

For massive particles we can choose the rest frame

$$P_\mu = (m, 0, 0, 0)$$

as the reference frame to build the supermultiplets.

First of all we will discuss the simple supersymmetric case and then we will consider the extended supersymmetric scenario.

- Massive multiplets in $N = 1$.

The susy algebra becomes

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2m\, \sigma^0_{\alpha\dot\beta} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case none of the generators vanish on the representation, so we have two pairs of raising and lowering operators. However, in the massive case, the Q_α and $\bar{Q}_{\dot\alpha}$ operators act on massive Lorentz representation with a given spin j giving new states with spin $j \pm 1/2$:

$$Q_\alpha |j\rangle \Rightarrow |j+1/2\rangle \text{ and } |j-1/2\rangle;$$
$$\bar{Q}_{\dot\alpha} |j\rangle \Rightarrow |j+1/2\rangle \text{ and } |j-1/2\rangle.$$

NOTE. Obviously if $j=0$ we only have $Q_\alpha|0\rangle \Rightarrow |1/2\rangle$, and analogously for $\bar{Q}_{\dot\alpha}$.

We can define normalized raising and lowering operators

$$\begin{cases} a_\alpha \equiv \dfrac{1}{\sqrt{2m}} Q_\alpha \\ a^+_{\dot\alpha} = \dfrac{1}{\sqrt{2m}} \bar{Q}_{\dot\alpha} \end{cases},$$

which satisfy the anticommutation relations

$$\begin{cases} \{a_\alpha, a^+_\beta\} = \delta_{\alpha\beta} \\ \{a_\alpha, a_\beta\} = \{a^+_\alpha, a^+_\beta\} = 0 \end{cases}.$$

To build the representations we start from the Clifford vacuum $|\Omega_j\rangle$ which is defined by

$$a_\alpha |\Omega_j\rangle = 0 \qquad \alpha = 1, 2.$$

Using the algebra of the a_α generators one can prove that such a state always exists in a representation (exercise).

In this case $|\Omega_j\rangle$ is a massive Lorentz representation, this means that it is a state with a given spin j and has degeneracy $2j+1$ since j_3 takes the values $-j, \dots, j$. Starting from Clifford vacua with different j we find different susy representations.

For a given $|\Omega_s\rangle$ the full massive susy representation is

$$|\Omega_s\rangle$$
$$a_1^+|\Omega_s\rangle$$
$$a_2^+|\Omega_s\rangle$$
$$\frac{1}{\sqrt{2}}a_1^+a_2^+|\Omega_s\rangle = -\frac{1}{\sqrt{2}}a_2^+a_1^+|\Omega_s\rangle .$$

There are a total of $4\cdot(2j+1)$ states in this representation.

The spin of the states is

$$|\Omega_s\rangle \quad , \quad \frac{1}{\sqrt{2}}a_1^+a_2^+|\Omega_s\rangle \qquad \Rightarrow \text{ spin } j$$

$$a_1^+|\Omega_s\rangle, \; a_2^+|\Omega_s\rangle \qquad \Rightarrow \text{ spin } j+1/2 \text{ and } j-1/2$$

($j-1/2$ is there only for $j \geq 1/2$)

To prove that $\frac{1}{\sqrt{2}}a_1^+a_2^+|\Omega_s\rangle$ has spin j we can use the equivalent representation

$$\frac{1}{\sqrt{2}}a_\alpha^+a_\beta^+|\Omega_s\rangle = \frac{1}{2\sqrt{2}}\varepsilon_{\alpha\beta}(a^\gamma)^+a_\gamma^+|\Omega_s\rangle .$$

One can show that the $(a^\gamma)^+a_\gamma^+$ operator is rotationally invariant, so it has spin zero. By applying it to $|\Omega_s\rangle$ we again get a state of spin j.

- If the Clifford vacuum has <u>spin zero</u> (<u>$j=0$</u>) the multiplet is given by
  - two states of spin 0 (bosons)
  - a state of spin $1/2$ (fermion)

- If the Clifford vacuum has <u>spin $j>0$</u> the multiplet contains
  - two states of spin j
  - one state of spin $j-1/2$
  - one state of spin $j+1/2$

· Massive multiplets in $N>1$.

For extended susy we have two possible situations depending on whether the central charges vanish or not.

· No central charges

This case is very similar to the simple susy case. We have $2N$ pairs of raising and lowering operators

$$\begin{cases} a^I_\alpha = \dfrac{1}{\sqrt{2m}} Q^I_\alpha \\ (a^I_\alpha)^+ = \dfrac{1}{\sqrt{2m}} \bar{Q}^I_{\dot\alpha} \end{cases},$$

which satisfy the anticommutation relations

$$\begin{cases} \{a^I_\alpha, (a^J_\beta)^+\} = \delta_{\alpha\beta}\, \delta^{IJ} \\ \{a^I_\alpha, a^J_\beta\} = \{(a^I_\alpha)^+, (a^J_\beta)^+\} = 0 \end{cases}$$

This means that we have N copies of the operators of simple susy and we can generate the representations in a similar way. Starting from a Clifford vacuum with spin j we get a representation with $2^{2N}(2j+1)$ states.

The algebra of generators in this case exhibits an $SU(2)\times USp(2N)$ symmetry. This can be seen by defining the new set of operators

$$\begin{cases} q^\ell_\alpha = a^\ell_\alpha \\ q^{N+\ell}_\alpha = \sum\limits_{\beta=1}^{2} \varepsilon_{\alpha\beta}\, (a^\ell_\beta)^+ \end{cases} \qquad \ell = 1, .., N.$$

Under hermitian conjugation

$$(q^r_\alpha)^+ = \varepsilon^{\alpha\beta}\, \Lambda^{rt}\, q^t_\beta \,,$$

where $r,t = 1, \dots, 2N$ and

$$\Lambda = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right).$$

The anticommutation relations of the q's are

$$\{q^r_\alpha, q^t_\beta\} = -\varepsilon_{\alpha\beta}\, \Lambda^{rt}.$$

This shows the invariance under $SU(2)$ (which acts on the α and β indices) and $USp(2N)$ (which acts on the r and t indices). This invariance group is useful because states of a given spin form irreducible representations of $USp(2N)$.

NOTE. The $SU(2)\times USp(2N)$ symmetry is actually a subgroup of a larger symmetry group of the algebra: $SU(2)\times USp(2N) \subset SO(4N)$. (see for example Wess, Bagger).

· <u>Non-vanishing central charges.</u>

Now we consider the case in which central charges are non-zero. The susy algebra with $P_\mu = (m,0,0,0)$ is

$$\begin{cases} \{Q^I_\alpha, (Q^J_\beta)^+\} = 2m\, \delta_{\alpha\beta}\, \delta^{IJ} \\ \{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta} Z^{IJ} \\ \{(Q^I_\alpha)^+, (Q^J_\beta)^+\} = \varepsilon_{\alpha\beta} (Z^{IJ})^* \end{cases}$$

To simplify the algebra we can use an appropriate $U(N)$ rotation of the Q's which brings the antisymmetric Z^{IJ} matrix into a standard form

$$Z^{IJ} = \begin{pmatrix} 0 & q_1 & & 0 & \\ -q_1 & 0 & & & \cdots \\ & & 0 & q_2 & \\ 0 & & -q_2 & 0 & \\ & & \vdots & & \end{pmatrix} \qquad \text{with} \quad q_n \geq 0$$

with a last row and column of zeroes in the case of odd N.

If N is odd we will have the N-th operator which satisfies

$$\begin{cases} \{Q^N_\alpha, (Q^N_\beta)^+\} = 2m\, \delta_{\alpha\beta} \\ \{Q^N_\alpha, Q^N_\beta\} = \{(Q^N_\alpha)^+, (Q^N_\beta)^+\} = 0 \end{cases}$$

so we have a pair of raising and lowering operators

$$\begin{cases} a^N_\alpha \equiv \dfrac{1}{\sqrt{2m}}\, Q^N_\alpha \\ (a^N_\alpha)^+ \equiv \dfrac{1}{\sqrt{2m}}\, \bar{Q}^N_\alpha \end{cases}$$

These can be treated as in the case of vanishing central charges, so in the following we will focus on the case with <u>N even</u>.

In this case we can define $2N$ pairs of raising and lowering operators

$$a^L_\alpha = \frac{1}{\sqrt{2}} \left(Q^{2L-1}_\alpha + \varepsilon_{\alpha\beta}\, \bar{Q}^{2L\dot\beta} \right)$$

$$b^L_\alpha = \frac{1}{\sqrt{2}} \left(Q^{2L-1}_\alpha - \varepsilon_{\alpha\beta}\, \bar{Q}^{2L\dot\beta} \right) \qquad L = 1, \ldots, N/2,$$

and their hermitian conjugates $(a^L_\alpha)^+$, $(b^L_\alpha)^+$.

Notice that the Lorentz structure is ruined, but the important point is that Q_α and $\bar{Q}^{\dot\alpha}$ transform in the same way under <u>spatial</u> rotations. This means that $(a^L_\alpha)^+$ and $(b^L_\alpha)^+$ create states of definite spin.

The anticommutation relations for the a^L_α and b^L_α are

$$\{a^r_\alpha, (a^s_\beta)^+\} = (2m - q_r)\, \delta_{rs}\, \delta_{\alpha\beta},$$

$$\{b^r_\alpha, (b^s_\beta)^+\} = (2m + q_r)\, \delta_{rs}\, \delta_{\alpha\beta},$$

$$\{a^r_\alpha, (b^s_\beta)^+\} = \{a^r_\alpha, a^s_\beta\} = \cdots = 0 .$$

Positivity of the Hilbert space implies that

$$2m \geq |q_n| \qquad \forall n .$$

NOTE. This relation gives another proof of the fact that for massless representations all central charges must be trivially represented, that is they must vanish on the representation states.

If some (or all) of the q_n saturate the bound (i.e. $|q_n| = 2m$), then the corresponding operators must be set to zero, as we did in the massless case with the Q_2^I.

- When $2m > |q_n|$ for all n, the multiplicities of the massive irreducible representations are the same as for the case of no central charges. A multiplet will contain $2^N(2j+1)$ states.

- When some of the bounds are saturated we loose some of the creation operators. If r central charges saturate the bound we are left with a Clifford algebra of $2(N-r)$ raising and lowering operators. The corresponding representations are similar to the ones for the case without central charges and with N reduced by r.

- When no central charge bound is saturated we get multiplets called "long multiplets". If some of the bounds are saturated we have a "short multiplet".

As an example let us compare the long and short representations for the $N=2$ case. For $q < 2m$ we have the long multiplet with $|\Omega\rangle$ of spin 0:

spin	spin reps.	number of states
0	5	5
1/2	4	8
1	1	3

for $q = 2m$ we have the short multiplet

spin	spin reps.	number of states
0	2	2
1/2	1	2
1	0	0

as one can check the short multiplet has the same number of states as the $N=2$ massless hypermultiplet.

For a Clifford vacuum with $j=1/2$ we get the <u>long multiplet</u>

spin	spin reps.	number of states
0	4	4
1/2	6	12
1	4	12
3/2	1	4

for the <u>short multiplet</u>

spin	spin reps.	number of states
0	1	1
1/2	2	4
1	1	3
3/2	0	0

Again the short multiplet has the same degrees of freedom as the massless vector multiplet.

NOTE. The states which saturate the bounds and lead to short multiplets are also called <u>BPS - states</u>. This is because of their analogy with the BPS monopoles in gauge theories (BPS comes from Bogomolnyi, Prasad, Sommerfeld).

The states which saturate some BPS bounds are also called "supersymmetric states". This comes from the fact that they are invariant under a part of the susy algebra (for the operators which saturate the bounds $(Q^I_\alpha)^+|BPS\rangle = 0$). For example a state which saturates all the bounds preserves $1/2$ of the supersymmetry.

A short multiplet is "stable" under radiative corrections. This means that its mass can not be changed by radiative corrections. The reason is simple: if the BPS bound is not satisfied any more, the multiplet should have more states than the short multiplet, but this is not possible because (small) perturbations can not change a discrete quantity like the number of states in a multiplet.

This is a strong property which follows from the fact that we related a physical quantity (the mass) to the symmetry algebra of the theory. Moreover this is an example of the "protection mechanisms" which are provided by supersymmetry.

· The $USp(2N)$ group.

We have seen that in the case of massive representations without central charges we can define the following set of operators

$$\begin{cases} q_\alpha^{\ell} = a_\alpha^{\ell} \\ q_\alpha^{N+\ell} = \sum_{\beta=1}^{2} \varepsilon_{\alpha\beta}\,(a_\beta^{\ell})^+ \end{cases} \qquad \ell = 1, .., N$$

which satisfy the relations

$$(q_\alpha^{r})^+ = \varepsilon^{\alpha\beta}\,\Lambda^{rt}\,q_\beta^{t} \qquad (*)$$

and

$$\{q_\alpha^{r}, q_\beta^{t}\} = -\varepsilon_{\alpha\beta}\,\Lambda^{rt}$$

where

$$\Lambda = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right)_{2N\times 2N} .$$

· One can easily check that taken $U_{\alpha\beta} \in SU(2)$, the algebra is invariant under

$$q_\alpha^{r} \to U_{\alpha\beta}\,q_\beta^{r} \equiv q'^{r}_\alpha$$

This follows from

$$\{q'^{r}_\alpha, q'^{t}_\beta\} = \{U_{\alpha\gamma}q_\gamma^{r}, U_{\beta\delta}q_\delta^{t}\} = -U_{\alpha\gamma}U_{\beta\delta}\,\varepsilon_{\gamma\delta}\,\Lambda^{rt}$$

$$= -U_{\alpha\gamma}\,\varepsilon_{\gamma\delta}\,(U^T)_{\delta\beta}\,\Lambda^{rt} = -(U\varepsilon U^T)_{\alpha\beta}\,\Lambda^{rt} = -\varepsilon_{\alpha\beta}\,\Lambda^{rt},$$

where we used that $\varepsilon = i\sigma^2$ and $\sigma^2 U^T = U^+\sigma^2$.

· Now we investigate what kind of invariance we have for the r and t indices. We can consider a transformation

$$q_\alpha^{r} \to q'^{r}_\alpha \equiv S^{rt}\,q_\alpha^{t} .$$

The algebra changes as

$$\{q'^{r}_\alpha, q'^{t}_\beta\} = \{S^{rm}q_\alpha^{m}, S^{tn}q_\beta^{n}\} = -\varepsilon_{\alpha\beta}\,S^{rm}S^{tn}\Lambda^{mn}$$

$$= -\varepsilon_{\alpha\beta}\,(S\Lambda S^T)^{rt},$$

to be invariant we need

$$S\Lambda S^T = \Lambda,$$

which is the definition of the symplectic group $Sp(2N)$.

But we must also respect the relation $(*)$, which imposes a "reality" condition which leads to the group $\underline{USp(2N)}$. To see this we apply the $Sp(2N)$ transformation to $(*)$:

$$(S^{rm}q_\alpha^{m})^+ = \varepsilon^{\alpha\beta}\Lambda^{rt}S^{tm}q_\beta^{m}.$$

From this we get

$$(S^{+T})^{rm}(q_\alpha^{\hat{r}})^{+} = \varepsilon^{\alpha\beta}\Lambda^{rt}S^{tm}q_\beta^{m}$$

$$\Rightarrow (q_\alpha^{n})^{+} = ((S^{-1})^{+T})^{nr}\varepsilon^{\alpha\beta}\Lambda^{rt}S^{tm}q_\beta^{m}$$

$$= ((S^{-1})^{+T}\Lambda S)^{nm}\varepsilon^{\alpha\beta}q_\beta^{m}$$

To reproduce (*) we need

$$\underline{\Lambda = (S^{-1})^{+T}\Lambda S}. \qquad (**)$$

Using the relation $\Lambda^{-1} = -\Lambda$, we can manipulate the constraint $S\Lambda S^T = \Lambda$:

$$-S\Lambda^{-1}S^T = -\Lambda^{-1}$$

$$\Rightarrow (S^{-1})^T\Lambda S^{-1} = \Lambda$$

$$\Rightarrow \underline{\Lambda = S^T\Lambda S}.$$

Comparing this equation with (**) we get

$$S^T = (S^{-1})^{+T}$$

$$\Rightarrow \underline{S = (S^{-1})^{+}}$$

which tells us that S must be <u>unitary</u>, or, in other words, the symmetry group is <u>$USp(2N)$</u>.

<u>NOTE</u>. An important point in this discussion is the fact that we can mix the Q_α^I generators with the $\bar{Q}^{I\dot\alpha}$ generators to build a single set of operators, like the ones we used to show the $USp(2N)$ symmetry. This is possible only because we have to respect only the <u>subgroup</u> of the Lorentz group which is unbroken in the representation. When we choose $P_\mu = (m,0,0,0)$ we break the Lorentz invariance to the $SU(2)$ subgroup of <u>spatial rotations</u> (the unbroken subgroup is usually called the "little group"), and we must preserve only this subgroup when we build the multiplets.

Q_α and $\bar{Q}^{\dot\alpha}$ transform in the same way under spatial rotations, so we can mix them in the algebra (this is also done for the case of massive representations with central charges).

In general the symmetry group of the algebra on a specific representation can be different from the symmetry group of the algebra itself. This is a consequence of the fact that some symmetries can be broken and some generators can take specific values (for example they can vanish) which allow for a different symmetry group.