Exercise 1) Invariant Measure on SU(2)

On a compact Lie group G, there exists a (up to scaling) unique left-invariant integration measure, the *Haar measure* dg:

$$\int_G f(g)dg = \int_G f(hg)dg$$

for all functions $f : G \to \mathbb{C}$ and all $h \in G$. The goal of this exercise is to find an expression for the Haar measure of $SU(2)$.

Find a parametrization of $SU(2)$, in which you can see that $SU(2)$ is homeomorphic to the 3-sphere $S^3 := \{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = 1\}$.

Now note that the Lebesgue measure $dxdydzdw$ on $\mathbb{R}^4$ is invariant under rotations. So if we change to polar coordinates

$$x = r\cos\theta\cos\phi$$
$$y = r\cos\theta\sin\phi$$
$$z = r\sin\theta\cos\chi$$
$$w = r\sin\theta\sin\chi\ ,$$

where $0 \leq \theta \leq \pi/2$, $0 \leq \phi$, $\chi \leq 2\pi$, then $dxdydzdw = |\det J|d\theta d\phi d\chi$, with the Jacobian

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \chi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \chi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \chi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} & \frac{\partial w}{\partial \chi} \end{pmatrix}$$

Calculate the Haar measure dg of $SU(2)$ in this parametrization with the normalization $\int_g dg = 1$.

Using your explicit formula for dg, show that

$$\int g|0\rangle\langle 0|g^\dagger dg = \mathbb{1}/2\ . \tag{1}$$

Exercise 2) Schur's Lemma

Show (1) using Schur's lemma.