Exercise 1) Clebsch-Gordan Coefficients

The goal of this exercise to calculate - for a second time - the Clebsch-Gordan coefficients. This time, the construction of the representations of $SU(2)$ as sub-representations of tensor products of the defining representation will be used.

a) V_k is a sub-representation of $V_1^{\otimes k}$ that appears with multiplicity one. Show that for $0 \leq l \leq k$:

$$|k,l\rangle = \frac{1}{\sqrt{\binom{k}{l}}} \left(|\underbrace{11\ldots1}_{l}\underbrace{00\ldots0}_{k-l}\rangle + \text{permutations} \right),$$

where $|1\rangle \equiv |1,1\rangle$ and $|0\rangle \equiv |1,0\rangle$.

b) Consider the decomposition $V_k \otimes V_1 \cong V_{k+1} \oplus V_{k-1}$ and recall that the weight of $|k,l\rangle$ is $2l-k$. Find the **bold** unknowns in the equation

$$|k+1,l\rangle = c_1|k,\mathbf{l_1}\rangle \otimes |1,1\rangle + c_2|k,\mathbf{l_2}\rangle \otimes |1,0\rangle$$

using the fact that the states on the RHS have the same weight as the state on the LHS. Calculate the coefficients c_1 and c_2 using the formula found in a).

c) Finally we want to express the weight states in V_{k-1} in terms of the basis according to the decomposition $V_k \otimes V_1 \cong V_{k+1} \oplus V_{k-1}$. For this, notice that $|k-1,l-1\rangle$ and $|k+1,l\rangle$ have the same weight in different irreducible representations. Use the orthogonality of these two states (they are in different irreducible representations) to obtain the decomposition of $|k-1,l-1\rangle$. [1]

Exercise 2) Representations of the Symmetric Group

Denote the symmetric group of degree n by S_n.

a) Write down the matrices of the representation of S_2 on $(\mathbb{C}^2)^{\otimes 2}$. Decompose this representation into a direct sum of irreducible representation and show explicitly how S_2 acts on the direct summands of this decomposition.

b) Write down the matrices of the representation of S_3 on $(\mathbb{C}^2)^{\otimes 3}$. Decompose this representation into a direct sum of irreducible representation and show explicitly how S_3 acts on the direct summands of this decomposition.

[1] There is also a way to calculate the states of V_{k-1} in a way similar to the way the states of V_{k+1} have been calculated in part b). This is done by considering representations of $U(2)$ instead of $SU(2)$. For more information see *Lie Algebras in Physics - From Isospin to Unified Theories* of Howard Georgi on pages 160-164 (in the recent 1999 version).