Repetition: Linear Algebra

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This is a brief reminder of some facts from linear algebra that will be needed in the course.

Theorem 1 (Spectral theorem)**.** *Every Hermitian operator $A : \mathbb{C}^d \to \mathbb{C}^d$ has an orthonormal basis $\{|v_i\rangle\}_{i=1}^d$ (its eigenbasis) and real numbers $\{\lambda_i\}_{i=1}^d$ with $\lambda_i \geq \lambda_{i+1}$ (its eigenvalues) such that*

$$A = \sum_{i=1}^{d} \lambda_i |v_i\rangle\langle v_i|.$$

In other words, there exists a unitary matrix U such that

$$A = UDU^\dagger$$

where $D = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$.

Proof. See e.g. [2] or [1]. □

Corollary 2 (Singular value decomposition)**.** *For every linear operator $A : \mathbb{C}^d \to \mathbb{C}^d$ there are two unitaries V and W and non-negative numbers s_i with $s_i \geq s_{i+1}$ (the singular values) such that*

$$A = VSW.$$

where $S = \mathrm{diag}(s_1, \dots, s_d)$.

Proof. We base our proof on the spectral theorem. $A^\dagger A$ is Hermitian since $(A^\dagger A)^\dagger = A^\dagger (A^\dagger)^\dagger = A^\dagger A$. By the spectral theorem, $A^\dagger A = UDU^\dagger$, for a unitary U and $D = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$ with $\lambda_i \geq \lambda_{i+1}$. Since $A^\dagger A$ is furthermore positive semi-definite ($\langle v| A^\dagger A |v\rangle = \langle w|w\rangle \geq 0$), all eigenvalues are non-negative, $\lambda_i \geq 0$ for all i.

Define $S := D^{\frac{1}{2}} = \mathrm{diag}(\lambda_1^{\frac{1}{2}}, \dots, \lambda_d^{\frac{1}{2}})$, and $\tilde{V} := AUS^{-1}$ where S^{-1} denotes the pseudo-inverse of S, here $S^{-1} = \mathrm{diag}(s_1^{-1}, \dots, s_k^{-1}, 0, \dots, 0)$, where k is the largest integer with $s_k > 0$. $\tilde{V}$ is a partial isometry [1] since

$$\tilde{V}^\dagger \tilde{V} = (AUS^{-1})^\dagger (AUS^{-1}) = S^{-1}U^\dagger A^\dagger AUS^{-1} = S^{-1}DS^{-1} = \sum_{i=1}^{k} |i\rangle\langle i|.$$

[1] An isometry $V : \mathbb{C}^d \to \mathbb{C}^{d'}$ is a matrix that satisfies $V^\dagger V = \mathbf{1}_{\mathbb{C}^d}$. A partial isometry is an isometry on a subspace of $\mathbb{C}^d$, i.e. $V^\dagger V = P$ for a projector P onto this subspace.

This implies that the first k columns of $\tilde{V}$ are orthonormal and the remaining empty. Let V be a unitary matrix whose first k columns are identical to $\tilde{V}$. Then $\tilde{V}S = VS$.

It is true that $\ker A^\dagger A = \ker A$. The inclusion $\subseteq$ is obvious. The opposite inclusion follows since for $|v\rangle \notin \ker A$, $A|v\rangle \neq 0$, hence $\langle v| A^\dagger A |v\rangle \neq 0$, hence $A^\dagger A |v\rangle \neq 0$ which implies $|v\rangle \notin \ker A^\dagger A$. The projector onto the complement of the kernel of A therefore takes the form $U \sum_{i=1}^k |i\rangle\langle i| U^\dagger$.

The statement of the corollary then follows for $W = U^\dagger$ since

$$VSW = \tilde{V}SW = AUS^{-1}SU^\dagger = AU \sum_{i=1}^k |i\rangle\langle i| U^\dagger = A$$

□

Note that the proof simplifies a little if $k = d$.

Lemma 3. *Let $\mathcal{H}_A \cong \mathbb{C}^d$ and $\mathcal{H}_B \cong \mathbb{C}^{d'}$ be finite dimensional complex vector spaces. Then we have a vector space isomorphism*

$$Hom(\mathcal{H}_A, \mathcal{H}_B) \cong \mathcal{H}_{A^*} \otimes \mathcal{H}_B$$

where $\mathcal{H}_{A^*}$ is the vector space dual to $\mathcal{H}_A$, given by

$$Hom(\mathcal{H}_A, \mathcal{H}_B) \ni K \mapsto \mathbf{1}_{A^*} \otimes K |\Phi\rangle_{A^*A} \in \mathcal{H}_{A^*} \otimes \mathcal{H}_B,$$

where $|\Phi\rangle_{A^*A} = \sum_{k=1}^d |k\rangle_{A^*} |k\rangle_A$ for an orthonormal basis $\{|k\rangle_A\}$ with dual basis $\{|k\rangle_{A^*}\}$.

Proof. $K = \sum_{ij} a_{ij} |i\rangle_B \langle j|_A$. Then

$$\mathbf{1}_{A^*} \otimes K |\Phi\rangle_{A^*A} = \sum_{i=1}^{d'} \sum_{j=1}^{d} \sum_{k=1}^{d} a_{ij} (\mathbf{1}_{A^*} \otimes |i\rangle_B \langle j|_A) |kk\rangle_{A^*A} = \sum_{i=1}^{d'} \sum_{k=1}^{d} a_{ij} |j\rangle_{A^*} |i\rangle_A .$$

□

Corollary 4 (Invariance of maximally entangled state)**.** *Let $A : \mathbb{C}^d \to \mathbb{C}^d$ be linear and let $|\Phi\rangle = \sum_{i=1}^d |ii\rangle$*

$$A \otimes \mathbf{1} |\Phi\rangle = \mathbf{1} \otimes A^T |\Phi\rangle$$

Proof.

$$\begin{aligned} A \otimes \mathbf{1} |\Phi\rangle &= \sum_{kl} a_{kl} |k\rangle \langle l| \otimes \mathbf{1} \sum_i |ii\rangle \\ &= \sum_{kli} a_{kl} |k\rangle \langle l|i\rangle |i\rangle \\ &= \sum_{kl} a_{kl} |k\rangle |l\rangle \end{aligned}$$

On the other hand

$$\begin{aligned}\mathbf{1} \otimes A^T |\Phi\rangle &= \sum_{kl} a_{kl} \mathbf{1} \otimes |l\rangle \langle k| \sum_i |ii\rangle \\ &= \sum_{kli} a_{kl} |i\rangle |l\rangle \langle k|i\rangle \\ &= \sum_{kl} a_{kl} |k\rangle |l\rangle\end{aligned}$$

This proves the claim. □

Corollary 5 (Schmidt decomposition)**.** *Let $|\phi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ (where we assume $\mathcal{H}_A \cong \mathcal{H}_B$), then there exist o.n. bases $\{|v_i\rangle_A\}$ and $\{|w_i\rangle_B\}$ and non-negative numbers s_i s.th.*

$$|\phi\rangle_{AB} = \sum_i s_i |v_i\rangle_A |w_i\rangle_B$$

Proof. Let $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ be o.n. bases for $\mathcal{H}_A$ and $\mathcal{H}_B$. Then

$$|\phi\rangle_{AB} = \sum_{ij} a_{ij} |i\rangle_A |j\rangle_B .$$

The singular value decomposition of the matrix A with entries $(A)_{ij} = a_{ij}$ reads $A = VSW$ for unitaries V and W and a non-negative diagonal matrix S. Let $|\Phi\rangle = \sum_i |ii\rangle$.

$$\begin{aligned}|\phi\rangle_{AB} &= A \otimes \mathbf{1} |\Phi\rangle \\ &= VSW \otimes \mathbf{1} |\Phi\rangle \\ &= VS \otimes W^T |\Phi\rangle \\ &= V \otimes W^T \cdot S \otimes \mathbf{1} |\Phi\rangle \\ &= V \otimes W^T \cdot \sum_i s_i |ii\rangle \\ &= \sum_i s_i |v_i\rangle |w_i\rangle\end{aligned}$$

where $|v_i\rangle = V|i\rangle$ and $|w_i\rangle = W^T |i\rangle$ □

References

[1] R. Bhatia, *Matrix analysis*, Graduate Texts in Mathematics, Springer, 1996.

[2] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge University Press, 1985.