Exercise 1) Bloch Vector

We can think of a quantum state ρ on $\mathbb{C}^2$ as a hermitian, positive semi-definite operator with unit trace. Choosing a basis, the linear operators on $\mathbb{C}^2$ are 2×2 matrices, i.e. $\rho \in \mathrm{M}_2(\mathbb{C})$. It can be checked easily that $\mathrm{M}_2(\mathbb{C})$ forms a 4-dim. vector space over the field of complex numbers. But since ρ is hermitian, ρ lies in a 4-dim. real subspace. This is because for any $A \in \mathrm{M}_2(\mathbb{C})$ hermitian we have that $(\alpha A)^\dagger = \overline{\alpha} A^\dagger = \overline{\alpha} A$ is hermitian if and only if $\alpha \in \mathbb{R}$. Choosing the Pauli matrices, together with the unit $\mathbb{1}_2$ as a basis, we have

$$\rho \in \{a_0 \cdot \mathbb{1}_2 + a_1 \cdot \sigma_X + a_2 \cdot \sigma_Y + a_3 \cdot \sigma_Z | a_i \in \mathbb{R}\} .$$

Since $\mathrm{tr}(\sigma_i) = 0$ it follows from $\mathrm{tr}(\rho) = 1$ that $\mathrm{tr}(\rho) = 2a_0$ and hence $a_0 = \frac{1}{2}$. Since ρ is positive semi-definite it follows that the eigenvalues k_i of ρ have to be greater or equal than zero, i.e.

$$k_i = \frac{1}{2}[\mathrm{tr}(\rho) \pm \sqrt{\mathrm{tr}(\rho)^2 - 4\det(\rho)}] = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \det(\rho)} \geq 0 .$$

and hence $0 \leq \det(\rho) \leq \frac{1}{4}$. Now write $\rho = \begin{pmatrix} \frac{1}{2} + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & \frac{1}{2} - a_3 \end{pmatrix}$ and hence

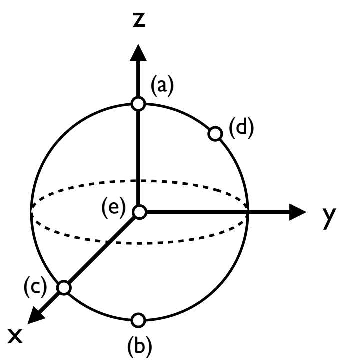
$$\det(\rho) = (\frac{1}{2} + a_3)(\frac{1}{2} - a_3) - (a_1 + ia_2)(a_1 - ia_2) = \frac{1}{4} - a_3^2 - a_1^2 - a_2^2 .$$

This gives us

$$k_i = \frac{1}{2} \pm \sqrt{a_1^2 + a_2^2 + a_3^2} = \frac{1}{2} \pm \sqrt{\frac{1}{4}(r_1^2 + r_2^2 + r_3^2)} \geq 0$$

for $\frac{1}{2} r_i := a_i$ and it follows that $|\vec{r}|^2 = \sqrt{r_1^2 + r_2^2 + r_3^2} \leq 1$ as well as $\rho = \frac{1}{2}(\mathbb{1}_2 + \vec{r} \cdot \vec{\sigma})$.

Graphical representation, where (a) corresponds to $|0\rangle$, (b) is $|1\rangle$, (c) is $|+\rangle$, (d) is $|-\rangle$ and (e) represents $\frac{\mathbb{1}}{2}$:



Exercise 2) Average State

1. Every density matrix ρ is hermitian and thus there exists a orthonormal basis $\{|i\rangle\}_{i\in I}$ in which ρ is diagonal (by the spectral decomposition theorem). Now let $P_i = |i\rangle\langle i|$ be the projector onto the one dimensional eigenstate $|i\rangle$ to eigenvalue k_i of ρ. Then $\rho = \sum_{i\in I} k_i |i\rangle\langle i|$ and if all the p_i are distinguishable, the spectral decomposition is unique and hence $p_i = k_i$ as well as $|\phi_i\rangle = |i\rangle$.
If some eigenvalues are identical, i.e. the characteristic polynomial has multiplicities, the decomposition is not unique (the partial orthonormal basis in the multiplicity space can be transformed by unitary transformations).

2. If the $\{|\Phi_i\rangle\}$ do not form an orthonormal basis, then the decomposition is unique if and only if the state ρ is pure.

Exercise 3) Partial Trace

The reduced density operator ρ_A is given by

$$\rho_A = \sum_{kl} |k\rangle\langle l|_A \mathrm{tr}[(|l\rangle\langle k| \otimes \mathbb{1}_B)(\sum_{klmn} \rho_{klmn}|k\rangle\langle l|_A \otimes |m\rangle\langle n|_B)] = \sum_{klm} \rho_{klmm}|k\rangle\langle l|_A \ .$$

Hermiticity of ρ_A follows from

$$\rho_A^\dagger = (\sum_{klm} \rho_{klmm}|k\rangle\langle l|_A)^\dagger = \sum_{klm} \overline{\rho}_{klmm}(|k\rangle\langle l|_A)^\dagger = \sum_{klm} \rho_{lkmm}|l\rangle\langle k|_A = \rho_A \ .$$

Since $\rho_{AB} \geq 0$, its scalar product with any pure state is non-negative. Now consider the pure states $|\Phi_i\rangle_{AB} = |\psi\rangle_A \otimes |i\rangle_B$ in $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$\begin{aligned}
0 &\leq \sum_i \langle\Phi_i|\rho_{AB}|\Phi_i\rangle = \sum_i (\langle\psi|_A \otimes \langle i|_B)\rho_{AB}(|\psi\rangle_A \otimes |i\rangle_B) \\
&= \sum_i (\langle\psi|_A \otimes \langle i|_B)(\sum_{klmn} \rho_{klmn}|k\rangle\langle l|_A \otimes |m\rangle\langle n|_B)(|\psi\rangle_A \otimes |i\rangle_B) \\
&= \sum_{iklmn} \rho_{klmn}\langle\psi|k\rangle\langle l|\psi\rangle_A\langle i|m\rangle\langle n|i\rangle_B \\
&= \sum_{mkl} \rho_{klmm}\langle\psi|k\rangle\langle l|\psi\rangle_A = \langle\psi|\rho_A|\psi\rangle_A \ .
\end{aligned}$$

Since this hold for any $|\psi\rangle_A \in \mathcal{H}_A$, it follows that ρ_A is positive semi-definite.

The normalization follows from

$$\mathrm{tr}(\rho_A) = \sum_n \langle n|_A(\sum_{mkl} \rho_{klmm}|k\rangle\langle l|_A)|n\rangle_A = \sum_{mn} \rho_{nnmm} = \mathrm{tr}(\rho_{AB}) = 1 \ .$$

The reduced density matrix of $\rho_{AB} = |\phi\rangle\langle\phi|_{AB}$ is mixed, even though $|\phi\rangle_{AB}$ is pure:

$$\begin{aligned}
\rho_{AB} &= \frac{1}{2}(r|00\rangle\langle 00|_{AB} + r\sqrt{1-r}|00\rangle\langle 11|_{AB} + r\sqrt{1-r}|11\rangle\langle 00|_{AB} + (1-r)|11\rangle\langle 11|_{AB}) \\
\rho_A &= \mathrm{tr}_B(\rho_{AB}) = \frac{1}{2}(r|0\rangle\langle 0|_A + (1-r)|1\rangle\langle 1|_A) \ .
\end{aligned}$$

The corresponding Bloch vector is given by $\vec{v}_r = (0, 0, 2r - 1)^T$.

We have $P_X(x) = \sum_y P_{XY}(x, y)$ with $P_X(x) \geq 0$ for all x and $\sum_x P_X(x) = \sum_{xy} P_{XY}(x, y) = 1$. Likewise for $P_Y(y)$.

The probability distribution P_{XY} can be represented as a quantum state using an orthonormal basis $\{|x\rangle_X \otimes |y\rangle_Y\}$: $P_{XY} \hat{=} \rho_{XY} = \sum_{xy} P_{XY}(x, y)|x\rangle\langle x|_X \otimes |y\rangle\langle y|_Y$.

For the partial trace over Y we get

$$\begin{aligned}\rho_X = \mathrm{tr}_Y(\rho_{XY}) &= \sum_{y'}(\mathbb{1}_X \otimes \langle y'|_Y)\rho_{XY}(\mathbb{1}_X \otimes \langle y'|_Y) = \sum_{xyy'} P_{XY}(x, y)|\langle y'|y\rangle|^2|x\rangle\langle x|_X \\ &= \sum_{xy} P_{XY}(x, y)|x\rangle\langle x|_X = \sum_x P_X(x)|x\rangle\langle x|_X \ ,\end{aligned}$$

which is a quantum representation of P_X. Likewise for the partial trace over X.