Exercise 1) Rotations on the Bloch sphere

Using a Taylor expansion it can be checked easily that

$$\exp(\pm i\frac{\alpha}{2}\vec{e}\cdot\vec{\sigma}) = \cos(\frac{\alpha}{2})\mathbb{1} \pm i(\vec{e}\cdot\vec{\sigma})\sin(\frac{\alpha}{2}) \ .$$

Furthermore it is straightforward to check that

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{a}\cdot\vec{c}) - \vec{c}\cdot(\vec{a}\cdot\vec{b})$$
$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\mathbb{1} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \ .$$

Now we calculate

$$\begin{aligned} U(\vec{e},\alpha)(\vec{v}\cdot\vec{\sigma})U(\vec{e},\alpha)^\dagger &= (\cos(\frac{\alpha}{2})\mathbb{1} - i\sin(\frac{\alpha}{2})\vec{e}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma})(\cos(\frac{\alpha}{2})\mathbb{1} + i\sin(\frac{\alpha}{2})\vec{e}\cdot\vec{\sigma}) \\ &= \cos^2(\frac{\alpha}{2})(\vec{v}\cdot\vec{\sigma}) + \sin^2(\frac{\alpha}{2})(\vec{e}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma})(\vec{e}\cdot\vec{\sigma}) \\ &+ i\cos(\frac{\alpha}{2})\sin(\frac{\alpha}{2})((\vec{v}\cdot\vec{\sigma})(\vec{e}\cdot\vec{\sigma}) - (\vec{e}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma})) \ . \end{aligned}$$

To evaluate the second term we calculate

$$\begin{aligned} (\vec{e}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma})(\vec{e}\cdot\vec{\sigma}) &= ((\vec{e}\cdot\vec{v})\mathbb{1} + i\vec{\sigma}\cdot(\vec{e}\times\vec{v}))(\vec{e}\cdot\vec{\sigma}) = (\vec{e}\cdot\vec{v})(\vec{e}\cdot\vec{\sigma}) + i(\vec{\sigma}\cdot(\vec{e}\times\vec{v}))(\vec{e}\cdot\vec{\sigma}) \\ &= (\vec{e}\cdot\vec{v})(\vec{e}\cdot\vec{\sigma}) + i(((\vec{e}\times\vec{v})\cdot\vec{e})\mathbb{1} + i\vec{\sigma}\cdot((\vec{e}\times\vec{v})\times\vec{e})) \\ &= (\vec{e}\cdot\vec{v})(\vec{e}\cdot\vec{\sigma}) - \vec{\sigma}\cdot((\vec{e}\times\vec{v})\times\vec{e}) = ((\vec{e}\cdot\vec{v})\vec{e} + (\vec{e}\times(\vec{e}\times\vec{v})))\cdot\vec{\sigma} \\ &= ((\vec{e}\cdot\vec{v})\vec{e} + \vec{e}(\vec{e}\cdot\vec{v}) - \vec{v}(\vec{e}\cdot\vec{e}))\cdot\vec{\sigma} = (2(\vec{e}\cdot\vec{v})\vec{e} - \vec{v})\cdot\vec{\sigma} \ . \end{aligned}$$

For the third term we can get

$$\begin{aligned} (\vec{v}\cdot\vec{\sigma})(\vec{e}\cdot\vec{\sigma}) - (\vec{e}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma}) &= (\vec{v}\cdot\vec{e})\mathbb{1} + i\vec{\sigma}\cdot(\vec{v}\times\vec{e}) - ((\vec{e}\cdot\vec{v})\mathbb{1} + i\vec{\sigma}\cdot(\vec{e}\times\vec{v})) \\ &= -2i(\vec{e}\times\vec{v})\cdot\vec{\sigma} \ . \end{aligned}$$

Putting everything together we get

$$\begin{aligned} U(\vec{e},\alpha)(\vec{v}\cdot\vec{\sigma})U(\vec{e},\alpha)^\dagger &= \cos^2(\frac{\alpha}{2})(\vec{v}\cdot\vec{\sigma}) + \sin^2(\frac{\alpha}{2})((2(\vec{e}\cdot\vec{v})\vec{e} - \vec{v})\cdot\vec{\sigma}) \\ &+ 2\cos(\frac{\alpha}{2})\sin(\frac{\alpha}{2})((\vec{e}\times\vec{v})\cdot\vec{\sigma}) \\ &= (\cos(\alpha)\vec{v} + \sin(\alpha)(\vec{e}\times\vec{v}) + (1-\cos(\alpha))(\vec{e}(\vec{e}\cdot\vec{v})))\cdot\vec{\sigma} \ . \end{aligned}$$

But by Rodrigues' rotation formula we have

$$R(\vec{e},\alpha)\vec{v} = \vec{v}\cos(\alpha) + (\vec{e}\times\vec{v})\sin(\alpha) + \vec{e}(\vec{e}\cdot\vec{v})(1-\cos(\alpha)) \ .$$

This concludes the proof.

Exercise 2) Quantum Teleportation

1. Before Bob receives the classical bit from Alice, the reduced density matrix on Bob's side can be calculated to $\mathbb{1}/2$ (i.e. he does not have any information about the state to transmit). Hence teleportation does not allow to transmit quantum states faster than light.
2. Since the teleportation protocol is linear and any mixed state input can be decomposed into its eigendecomposition $\rho = \sum_i p_i|i\rangle\langle i|$, the protocol also works for mixed states.

3. No, in general this is impossible because Alice does not know the state $|\psi\rangle$ of the qubit she has to send to Bob and the laws of quantum mechanics prevent her from determining the state when she only has a single copy of $|\psi\rangle$ in her possession. And even if she did know the state, describing it precisely takes an infinite amount of classical information since $|\psi\rangle$ takes values in a continuous space. So if she did know $|\psi\rangle$, it would take forever to send the state to Bob.

4. The crucial point is that only the target qubit at Bob's side is left in the state $|\psi\rangle$. The reduced density matrix on Alice's side after the protocol can be calculated to $|0\rangle$ or $|1\rangle$, depending upon her measurement result. That is, the information about the state $|\psi\rangle$ is no longer on Alice side, but it has been transferred to Bob's side.

Exercise 3) Entanglement Swapping

Notation: $|\phi^\pm\rangle := \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\psi^\pm\rangle := \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. Let's call the qubits of Alice and Bob that we wish to entangle A and B resp. and let them start in maximally entangles states with auxiliary qubits C_1 and C_2 at Charlie:

$$
\begin{aligned}
|\rho_{\text{in}}\rangle &= |\phi^+\rangle_{AC_1} \otimes |\phi^+\rangle_{C_2B} \\
&= \frac{1}{2}[|00\rangle_{AB} \otimes |00\rangle_{C_1C_2} + |01\rangle_{AB} \otimes |01\rangle_{C_1C_2} + |10\rangle_{AB} \otimes |10\rangle_{C_1C_2} + |11\rangle_{AB} \otimes |11\rangle_{C_1C_2}] \\
&= \frac{1}{2\sqrt{2}}[|00\rangle_{AB} \otimes (|\phi^+\rangle_{C_1C_2} + |\phi^-\rangle_{C_1C_2}) + |11\rangle_{AB} \otimes (|\phi^+\rangle_{C_1C_2} - |\phi^-\rangle_{C_1C_2}) \\
&+ |01\rangle_{AB} \otimes (|\psi^+\rangle_{C_1C_2} + |\psi^-\rangle_{C_1C_2}) + |10\rangle_{AB} \otimes (|\psi^+\rangle_{C_1C_2} - |\psi^-\rangle_{C_1C_2})] \\
&= \frac{1}{2}[|\phi^+\rangle_{AB} \otimes |\phi^+\rangle_{C_1C_2} + |\phi^-\rangle_{AB} \otimes |\phi^-\rangle_{C_1C_2} \\
&+ |\psi^+\rangle_{AB} \otimes |\psi^+\rangle_{C_1C_2} + |\psi^-\rangle_{AB} \otimes |\psi^-\rangle_{C_1C_2}] \ .
\end{aligned}
$$

Now Charlie can perform a projective measurement in the basis $\{|\phi^\pm\rangle, |\psi^\pm\rangle\}$ on his subsystem C_1C_2. Thereby the subsystem AB is projected in one of the maximally entangled states $\{|\phi^\pm\rangle, |\psi^\pm\rangle\}$, each with probability $\frac{1}{4}$. Afterwards Charlie sends the measurement outcome to Bob, who can then apply a local unitary to transform the state on AB into any desired maximally entangled state (as in the teleportation protocol).

One could also say, that we can teleport the state on AC_1 to Bob, using the ebit on C_2B.

This is a useful procedure to generate entanglement between two (possibly widely separated) systems. Note that there is no interaction between Alice and Bob.

Exercise 4) Representations of SU(2)

We need to show $[v_k(\sigma_i), v_k(\sigma_j)] = v_k[\sigma_i, \sigma_j]$ for all $i, j \in \{+, -, z\}$. Since $[\sigma_i, \sigma_j] = -[\sigma_j, \sigma_i]$, all commutators that we have to consider can be calculated to $[\sigma_+, \sigma_-] = \sigma_z$, $[\sigma_z, \sigma_+] = 2\sigma_+$ and $[\sigma_z, \sigma_-] = -2\sigma_-$. We calculate the first case and get

$$
v_k([\sigma_+, \sigma_-])|k, l\rangle = v_k(\sigma_z)|k, l\rangle = (2l - k)|k, l\rangle
$$

as well as

$$\begin{aligned}
[v_k(\sigma_+), v_k(\sigma_-)]|k,l\rangle &= v_k(\sigma_+)v_k(\sigma_-)|k,l\rangle - v_k(\sigma_-)v_k(\sigma_+)|k,l\rangle \\
&= v_k(\sigma_+)\sqrt{l(k-l+1)}|k,l-1\rangle - v_k(\sigma_-)\sqrt{(k-l)(l+1)}|k,l+1\rangle \\
&= \sqrt{l(k-l+1)}v_k(\sigma_+)|k,l-1\rangle - \sqrt{(k-l)(l+1)}v_k(\sigma_-)|k,l+1\rangle \\
&= l(k-l+1)|k,l\rangle - (k-l)(l+1)|k,l\rangle = (2l-k)|k,l\rangle\ .
\end{aligned}$$

Likewise for the other cases.

First notice that $\sigma_x = \sigma_+ + \sigma_-$ and $\sigma_y = i(\sigma_- - \sigma_+)$. This implies

$$\begin{aligned}
[\sum_i v_k(\sigma_i)v_k(\sigma_i)]|k,l\rangle &= [(v_k(\sigma_x))^2 + (v_k(\sigma_y))^2 + (v_k(\sigma_z))^2]|k,l\rangle \\
&= [(v_k(\sigma_+ + \sigma_-))^2 + (v_k(i\sigma_- - i\sigma_+))^2 + (v_k(\sigma_z))^2]|k,l\rangle \\
&= [(v_k(\sigma_+) + v_k(\sigma_-))^2 - (v_k(\sigma_-) - v_k(\sigma_+))^2 + (v_k(\sigma_z))^2]|k,l\rangle \\
&= [2v_k(\sigma_+)v_k(\sigma_-) + 2v_k(\sigma_-)v_k(\sigma_+) + v_k(\sigma_z)v_k(\sigma_z)]|k,l\rangle \\
&= 2v_k(\sigma_+)v_k(\sigma_-)|k,l\rangle + 2v_k(\sigma_-)v_k(\sigma_+)|k,l\rangle + v_k(\sigma_z)v_k(\sigma_z)|k,l\rangle \\
&= k(k+2)|k,l\rangle\ .
\end{aligned}$$

Proof by induction. We have $|k+1,k+1\rangle = |k,k\rangle|1,1\rangle$, and applying the lowering operator $v_{k+1}(\sigma_-)$ to this we get

$$|k+1,k\rangle = \sqrt{\frac{k}{k+1}}|k,k-1\rangle|1,1\rangle + \sqrt{\frac{1}{k+1}}|k,k\rangle|1,0\rangle\ .$$

This is the basic step. Now assume that the claim holds for some l, that is

$$|k+1,l\rangle = \sqrt{\frac{l}{k+1}}|k,l-1\rangle|1,1\rangle + \sqrt{\frac{k+1-l}{k+1}}|k,l\rangle|1,0\rangle.$$

The inductive step is done by applying $v_k(\sigma_-)$ to both sides of this equation. For the LHS we get

$$v_k(\sigma_-)|k+1,l\rangle = \sqrt{l(k-l+2)}|k+1,l-1\rangle\ .$$

For the RHS we get

$$\begin{aligned}
&v_k(\sigma_-)(\sqrt{\frac{l}{k+1}}|k,l-1\rangle|1,1\rangle + \sqrt{\frac{k+1-l}{k+1}}|k,l\rangle|1,0\rangle) \\
&= \sqrt{\frac{l}{k+1}}v_k(\sigma_-)(|k,l-1\rangle|1,1\rangle) + \sqrt{\frac{k+1-l}{k+1}}v_k(\sigma_-)(|k,l\rangle|1,0\rangle) \\
&= \sqrt{\frac{l}{k+1}}((v_k(\sigma_-)|k,l-1\rangle)|1,1\rangle + |k,l-1\rangle(v_k(\sigma_-)|1,1\rangle)) \\
&+ \sqrt{\frac{k+1-l}{k+1}}(v_k(\sigma_-)|k,l\rangle)|1,0\rangle \\
&= \sqrt{\frac{l}{k+1}}\sqrt{(l-1)(k-l+2)}|k,l-2\rangle|1,1\rangle + \sqrt{\frac{l}{k+1}}|k,l-1\rangle|1,0\rangle \\
&+ \sqrt{\frac{k+1-l}{k+1}}\sqrt{l(k-l+1)}|k,l-1\rangle|1,0\rangle\ .
\end{aligned}$$

Combining this we get

$$|k+1,l-1\rangle = \sqrt{\frac{l-1}{k+1}}|k,l-2\rangle|1,1\rangle + \sqrt{\frac{k-l+2}{k+1}}|k,l-1\rangle|1,0\rangle\ .$$

This concludes the proof.